Elements of distribution theory

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1 Introduction

You have certainly encountered the Dirac delta function $\delta(x - x_0)$, which is often "defined" as a function that is zero everywhere except for the point x_0 where its value is infinity, in such a way that the integral $\int dx\, \delta(x - x_0)$ is equal to 1. This is definitely not a rigorous mathematical definition, since such a function does not exist (if a function is zero everywhere except in a point, its integral is zero).One may think: OK, this is physicist nonsense. Now, this is a bit extreme. Physicists may be sloppy in their language sometimes, but we will see that there is indeed a precise way to express the above statement.

What is clear is that the Dirac delta function is not a function in the proper sense, but it has to be defined in a more complicated way. To be precise, one should refer to the Dirac delta function as a *generalized function* or a *distribution*. Also, when we write something like

$$\int dx\, \delta(x - x_0) f(x) = f(x_0) \tag{1.1}$$

it is clear that the integral is not really an integral (since the integrand is not really a function). A proper understanding of distributions goes necessarily through a proper understanding of what the integral symbol means in this case. The theory of distributions is an extremely complicated, yet beautiful, branch of Mathematics. Here we want to provide only the main ideas.

2 Basic definitions

The basic logical step to understand distributions is to realized that the integral of a distribution times a function needs to be interpreted as a map from a space of functions to the complex numbers. For instance

$$\int dx\, \delta(x - x_0) f(x) = f(x_0) \tag{2.1}$$

should be more properly though as:

> *the application of the delta function centered in x_0* to the proper function $f(x)$ *gives the number $f(x_0)$.*

A better (i.e. more rigorous) way to denote this would probably be

$$\delta_{x_0}(f) = f(x_0) \ . \tag{2.2}$$

The definition of distributions goes in two steps: we first need to define the space of functions on which we want to apply the distributions, and then we can define the distribution themselves.

Let $\mathcal{S}$ be the space of **Schwartz functions**, i.e. functions $\phi(x)$ of a real variable x that are infinitely differentiable, and decay rapidly (i.e. faster than any inverse power of $|x|$) at infinity with all their derivatives. The property of rapid decay is equivalent to requiring that $|x^p \phi^{(q)}(x)|$ is a bounded function for every $p, q \in \mathbb{N}$, which we can write as

$$\|x^p \phi^{(q)}\|_\infty = \sup_{x \in \mathbb{R}} |x^p \phi^{(q)}(x)| < \infty \ . \tag{2.3}$$

In the context of the theory of distributions, $\mathcal{S}$ is called the set of **test functions**. It is easy to prove that $\mathcal{S}$ is a vector space, i.e. the sum of two Schwarz function is a Schwart function, and the multiplication of a Schwarz function times a constant is a Schwart function.

Example. The functions x^3 and $1/(1+x^2)$ are infinitely differentiable but they do not decay rapidly.

Example. The function $e^{-|x|}$ decays rapidly, but it is not differentiable in $x = 0$.

Example. The function e^{-x^2} is a Schwartz function.

We introduce a concept of convergence in the set of test functions. We will say that a sequence of test functions ϕ_n converges to zero in $\mathcal{S}$ if, for every $p, q \in \mathbb{N}$,

$$x^p \phi_n^{(q)}(x) \to 0 \qquad \text{as } n \to \infty \qquad \textit{uniformly,} \tag{2.4}$$

which is the same as saying that

$$\|x^p \phi_n^{(q)}\|_\infty \to 0 \ . \tag{2.5}$$

Also, we say that $\phi_n \to \phi$ if the difference sequance $\phi_n - \phi$ converges to zero.

Tempered distributions are defined as linear continuous functionals on $\mathcal{S}$. Let us see a bit more in detail what this means. First of all notice that we have used the word *tempered*, the reason being that there are other types of distributions associated to other test function spaces. Tempered distributions are among the most commonly used distributions in Physics. From now on we will omit the word *tempered*. Distributions are **functionals** on $\mathcal{S}$, i.e. they are maps $T : \mathcal{S} \to \mathbb{C}$ from the space of test functions into the complex numbers. They are **linear**, i.e. they satisfy

$$T(\alpha_1 \phi_1 + \alpha_2 \phi_2) = \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2) \ , \tag{2.6}$$

for all Schwartz functions ϕ_1, ϕ_2 and all complex numbers α_1, α_2. Finally, distributions are **continuous**, i.e.

$$T(\phi_n) \to 0 \qquad \text{whenever} \qquad \phi_n \to 0 \quad \text{in } \mathcal{S} \ . \tag{2.7}$$

The space of tempered distribution is denoted by $\mathcal{S}'$. Instead of the symbol $T(\phi)$ we will often use the notation $\langle T, \phi \rangle$ (but keep in mind that this is only notation, nothing more).

3 First example: regular distributions

Let $u(x)$ be a measurable function with the property that a positive integer N exists such that

$$M \equiv \int dx \frac{|u(x)|}{1 + x^{2N}} < \infty \ . \tag{3.1}$$

In particular we do not assume any regularity on the function $u(x)$. This class of functions includes e.g. polynomials, e^{ipx}, $\theta(x)$, $x^{-1/2}$, $\log |x|$ and much more horrible functions.

Then we define T_u the distribution associated to u as

$$T_u(\phi) = \int_{-\infty}^{\infty} dx \, u(x) \phi(x) = \langle T_u, \phi \rangle \ . \tag{3.2}$$

We want to see that this is indeed a distribution.

First we need to prove that for any test function ϕ, the above integral is finite. We use the inequality

$$|T_u(\phi)| \leq \int_{-\infty}^{\infty} dx\, |u(x)\phi(x)| = \int_{-\infty}^{\infty} dx\, \frac{|u(x)|}{1+x^{2N}} |(1+x^{2N})\phi(x)| \ . \tag{3.3}$$

In the above formula we have divided and multiplied the integrand by $1+x^{2N}$. Since ϕ is a Schwarts function, the function $(1+x^{2N})\phi(x)$ is bounded

$$\left|(1+x^{2N})\phi(x)\right| \leq |\phi(x)| + |x^{2N}\phi(x)| \leq \sup_{x\in\mathbb{R}} |\phi(x)| + \sup_{x\in\mathbb{R}} |x^{2N}\phi(x)| = \|\phi\|_\infty + \|x^{2N}\phi\|_\infty < \infty \ . \tag{3.4}$$

Therefore

$$|T_u(\phi)| \leq \left\{\|\phi\|_\infty + \|x^{2N}\phi\|_\infty\right\} \int_{-\infty}^{\infty} dx\, \frac{|u(x)|}{1+x^{2N}} = M\left\{\|\phi\|_\infty + \|x^{2N}\phi\|_\infty\right\} < \infty \tag{3.5}$$

which is finite since, by assumption, M is finite.

Then we have to prove that T_u is linear. This is obvious from the definition

$$\begin{aligned} T_u(\alpha\phi_1 + \beta\phi_2) &= \int_{-\infty}^{\infty} dx\, u(x)[\alpha\phi_1(x) + \beta\phi_2(x)] \\ &= \alpha \int_{-\infty}^{\infty} dx\, u(x)\phi_1(x) + \beta \int_{-\infty}^{\infty} dx\, u(x)\phi_2(x) \\ &= \alpha T_u(\phi_1) + \beta T_u(\phi_2) \ . \end{aligned} \tag{3.6}$$

Finally we need to prove that T_u is continuous. Let ϕ_n be a sequence that converges to zero in $\mathcal{S}$. We recycle the same inequality that we have used to prove the finiteness of the integral

$$|T_u(\phi_n)| \leq M\left\{\|\phi\|_\infty + \|x^{2N}\phi\|_\infty\right\} \ . \tag{3.7}$$

Since $\phi_n \to 0$ in $\mathcal{S}$, both $\|\phi_n\|_\infty$ and $\|x^{2N}\phi_n\|_\infty$ converge to zero (by definition of convergence in $\mathcal{S}$). Therefore $T_u(\phi_n) \to 0$ which proves continuity.

The distributions constructed in this way are called **regular distributions**. It is customary to identify the regular distribution T_u associated to u with u itself. One would therefore write

$$\langle u, \phi\rangle \equiv \langle T_u, \phi\rangle \ . \tag{3.8}$$

This is a slight abuse of notation which turns out to be very convenient. In this sense one says that the set of distributions $\mathcal{S}'$ includes all the functions u which satisfy the condition (3.1).

4 Second example: delta function

It should be clear at this point that the Dirac delta function can be defined simply as

$$\langle \delta_{x_0}, \phi\rangle = \phi(x_0) \ . \tag{4.1}$$

Both linearity and continuity are quite easy to prove. Give it a try!

5 Third example: principal value $1/x$

Another common distribution is the Cauchy principal value of $1/x$. This distribution is often denoted with one ov these symbols

$$\mathrm{P}\frac{1}{x} \ , \qquad \mathrm{PV}\frac{1}{x} \ , \qquad \frac{\mathrm{P}}{x} \ , \qquad \frac{\mathrm{PV}}{x} \ , \qquad \frac{\mathcal{P}}{x} \ . \tag{5.1}$$

Each of these symbols have to be considered as a single object, for instance there is no meaning associated with the PV and the $1/x$ separately. The Cauchy principal value of $1/x$ is defined, as any other distribution, by specifing its value when it acts on a test function. In this case,

$$\langle \frac{\mathrm{P}}{x}, \phi \rangle = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} dx\, \frac{\phi(x)}{x} = \lim_{\epsilon \to 0^+} \left\{ \int_{-\infty}^{-\epsilon} dx\, \frac{\phi(x)}{x} + \int_{\epsilon}^{\infty} dx\, \frac{\phi(x)}{x} \right\} \ . \tag{5.2}$$

First, we need to prove that the limit in the right-hand side is finite. This is not obvious, since the two terms in the curly brackets may diverge in the $\epsilon \to 0^+$ limit since the $1/x$ singularity at $x = 0$ is not integrable. The finiteness of the above limit comes from a cancellation of infinities.

To see how this works, we use the change of variables $x \to -x$ in the first integral in the curly brackets in eq. (5.2)

$$\langle \frac{\mathrm{P}}{x}, \phi \rangle = \lim_{\epsilon \to 0^+} \left\{ -\int_{\epsilon}^{\infty} dx\, \frac{\phi(-x)}{x} + \int_{\epsilon}^{\infty} dx\, \frac{\phi(x)}{x} \right\} = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} dx\, \frac{\phi(x) - \phi(-x)}{x} \ . \tag{5.3}$$

Now we observe that the integrand has a finite $x \to 0$ limit, as one can see e.g. by using de l'Hôpital

$$\lim_{x \to 0} \frac{\phi(x) - \phi(-x)}{x} = 2\phi'(0) \ . \tag{5.4}$$

Notice that in order to argue that this is finite, we need to use the fact that $\phi(x)$ is a Schwartz function, and in particular differentiable. Since the integrand in eq. (5.3) is continuous, the $\epsilon \to 0^+$ simply yields the integral over $x > 0$, i.e.

$$\langle \frac{\mathrm{P}}{x}, \phi \rangle = \int_{0}^{\infty} dx\, \frac{\phi(x) - \phi(-x)}{x} \ . \tag{5.5}$$

This show finiteness. Linearity is easy to prove, while continuity can be a little tricky, but give it a try!

6 Regularization of distributions

Let T_n be a sequence of distributions. We say that T_n converges to $T \in \mathcal{S}'$ if, for every test function ϕ,

$$\lim_{n \to \infty} \langle T_n, \phi \rangle = \langle T, \phi \rangle \ , \tag{6.1}$$

and we write also $T_n \to T$ as $n \to \infty$. This notion of convergence in $\mathcal{S}'$ is called **weak convergence**.

One can also consider a famililly of distributions T_ϵ which depends on a continuous parameter, typically $\epsilon > 0$. We will say that T_ϵ converges to $T \in \mathcal{S}'$ (or weakly) if, for every test function ϕ,

$$\lim_{\epsilon \to 0^+} \langle T_\epsilon, \phi \rangle = \langle T, \phi \rangle \ , \tag{6.2}$$

This concept of convergence is called weak because it "includes" other well known concepts of convergence, more precisely other types of convergenge usually imply weak convergence. Let us see a few examples.

1. If u_n is a sequence of continuous functions which converges uniformly to u, and if all these functions satisfy the condition (3.1), then T_{u_n} converges weakly to T_u (and we often just say u_n converges weakly to u). In fact

$$
\begin{aligned}
|\langle u_n, \phi \rangle - \langle u, \phi \rangle| &= \left| \int_{-\infty}^{\infty} dx \, [u_n(x) - u(x)] \phi(x) \right| \\
&\leq \int_{-\infty}^{\infty} dx \, |u_n(x) - u(x)| \, |\phi(x)| \\
&\leq \|u_n - u\|_\infty \int_{-\infty}^{\infty} dx \, |\phi(x)| \to 0 \ , \qquad (6.3)
\end{aligned}
$$

   where we notice that the integral of $|\phi(x)|$ is finite because $|\phi(x)|$ is contuous and decays rapidly at infinity, and $\|u_n - u\|_\infty \to 0$ by hypothesis of uniform convergence.

2. Let u_n be a sequence of function which converges to u pointwise (i.e. $u_n(x) \to u(x)$ for every x). If the sequence is dominated by a positive function w, i.e. $|u_n(x)| \leq w(x)$ for every n and x, and w satisfies the condition (3.1), then one can use Lebesgue's theorem of dominated convergence to show that $u_n \to u$ weakly.

3. Let u_n be a sequence of integrable functions which convergest to u in the sense of L^1. This means that

$$
\int_{\infty}^{\infty} dx \, |u_n(x) - u(x)| \to 0 \ . \qquad (6.4)
$$

   Then $u_n \to u$ weakly. This can be easily shown as follows

$$
\begin{aligned}
|\langle u_n, \phi \rangle - \langle u, \phi \rangle| &= \left| \int_{-\infty}^{\infty} dx \, [u_n(x) - u(x)] \phi(x) \right| \\
&\leq \int_{-\infty}^{\infty} dx \, |(x) - u(x)| \, |\phi(x)| \\
&\leq \|\phi\|_\infty \int_{-\infty}^{\infty} dx \, |u_n(x) - u(x)| \to 0 \ , \qquad (6.5)
\end{aligned}
$$

   where we have used the fact that every Schwartz function is bounded.

These are all examples that involve only regular distributions (i.e. proper functions). However the game becomes more interesting when we consider distributions that are not proper functions. Let us discuss a classical example.

The function

$$
u_\sigma(x) = \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} \qquad (6.6)
$$

is a regular distribution, i.e. it satisfies condition (3.1)*, for every* $\sigma > 0$*.* u_σ *converges to* δ_0 *in* $\mathcal{S}'$*, in the* $\sigma \to 0^+$ *limit.*

Let us prove the convergence. For every test function ϕ, we have

$$
\lim_{\sigma \to 0^+} \langle u_\sigma, \phi \rangle = \lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} dx \, \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} \phi(x) = \lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} dx \, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \phi(x\sigma^{1/2}) \ , \qquad (6.7)
$$

where we have used the change of variables $x \to x\sigma^{1/2}$. Let us assume for a moment that we can exchange the limit with the integral, then

$$\lim_{\sigma\to 0^+} \langle u_\sigma, \phi \rangle = \int_{-\infty}^{\infty} dx \, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \lim_{\sigma\to 0^+} \phi(x\sigma^{1/2}) = \phi(0) \int_{-\infty}^{\infty} dx \, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = \phi(0) = \langle \delta_0, \phi \rangle \ . \tag{6.8}$$

In fact the limit and integral can be exchanged thanks to Lebesgue's dominated convergence theorem, if one notices that

$$\left| \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \phi(x\sigma^{1/2}) \right| \leq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \|\phi\|_\infty \ , \tag{6.9}$$

and the dominating function is finite and integrable. ■

This simple exercise describes a way to approximate (in the sense of the weak limit) the delta function with a sequence or one-parameter family of regular distribution, or equivalently proper functions. This is very useful because the action of regular distributions is given by an integral of (proper) functions, which can be studied and undertood with stantard mathematical analysis tools. The above example motivates the following definition.

A sequence u_n of functions satisfying the condition (3.1) is said to be a **regularization** of the distribution T if u_n converges weakly to T. Analogously, a one-parameter family u_ϵ of functions satisfying the condition (3.1) is said to be a regularization of the distribution T if u_ϵ converges weakly to T in the $\epsilon \to 0^+$ limit.

The Gaussian function discussed in the above example is not the only regularization of the delta function. Other regularizations (with $\epsilon \to 0^+$ or $n \to \infty$) of δ_0 are for instance

$$\frac{\epsilon}{\pi(x^2+\epsilon^2)} \ , \qquad \frac{1}{\epsilon}\chi_{[0,\epsilon]}(x) \ , \qquad \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(x) \ , \qquad \frac{\sin(n\pi x)}{\pi x} \ . \tag{6.10}$$

One of the most interesting (and nontrivial) results in distribution theory is that **for every distribution there exists a regularization**, i.e. for every distribution T it is always possible to find a sequence of regular distributions u_n which converges weakly to T. Once again, weak convergence means that, for every test function ϕ,

$$\langle T, \phi \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx \, u_n(x)\phi(x) \ . \tag{6.11}$$

In a sense, this fact motivates the physicist notation

$$\langle T, \phi \rangle = \int_{-\infty}^{\infty} dx \, T(x)\phi(x) \qquad \textit{(sloppy physicist notation)} \ . \tag{6.12}$$

Once again, this is only notation since T is not a function and the integral here is not really an integral in the mathematical sense.

7 Derivatives of distributions

Given a distribution T we want to define the derivative of T, which we will denote by T' or DT. The minimal requirement that we want to fulfil, is that when T is a regular distribution associated to a differentiable bounded function u, i.e. $T = T_u$, then the derivative of T_u should be the regular distribution associated to u', i.e. we want that $DT_u = T_{u'}$. Now observe that

$$\langle T_{u'}, \phi \rangle = \int_{-\infty}^{\infty} dx\, u'(x)\phi(x) = -\int_{-\infty}^{\infty} dx\, u(x)\phi'(x) = -\langle T_u, \phi' \rangle \ . \tag{7.1}$$

We can use integration by parts here because both functions u and ϕ are differentiable, and the boundary term vanish beacuse ϕ decays rapidly at infinity. Therefore we define the **derivative** of a general distribution T by analogy as

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle \ , \tag{7.2}$$

or, which is the same with different notation,

$$T'(\phi) = -T(\phi') \ . \tag{7.3}$$

We should show that this definition makes sense, i.e. we should show that DT defined in this way is a distribution. First we notice that the right-hand side makes sense for any test function ϕ, because the derivative of a Schwartz function is still a Schwartz function (this is not completely obvious). Linearity of DT is a consequence of the linearity of T and of the derivative of functions. Finally continuity follows from the observation that, if $\phi_n \to 0$ in $\mathcal{S}$ then also $\phi'_n \to 0$ in $\mathcal{S}$, as one can easily prove from the definition of convergence in the set of test functions (sort out the details!). In this argument, we have assumed nothing special about T. In fact the derivative T' is defined (as a distribution) for every distribution T. In particular it follows that *every distribution is infinitely differentiable* and

$$\langle D^k T, \phi \rangle = (-1)^k \langle T, \phi^{(k)} \rangle \ . \tag{7.4}$$

In particular, every function u which satisfies the condition (3.1) (with no regularity assumption) is infinitely differentiable in the sense of distribution.

As an example let us prove that $\theta'(x) = \delta(x)$ where the derivative is taken in the sense of distributions. This follows from the chain of equalities

$$\langle D\theta, \phi \rangle = -\langle \theta, \phi' \rangle = -\int_{-\infty}^{\infty} dx\, \theta(x)\ \phi'(x) = -\int_{0}^{\infty} dx\, \phi'(x) = \phi(0) = \langle \delta_0, \phi \rangle \ . \tag{7.5}$$

where we have used the definition of derivative of distribution in the first equality, the definition of regular distributions in the second equality, the definition of $\theta(x)$ in the third equality, the fundamental theorem of algebra in the fourth equality (with the fact that ϕ is differentiable), and the definition of the distribution δ_0 in the last equality.

We point out two important results of distribution theory.

1. The derivative is a continuous operator on the space of distributions, i.e. if $T_n \to T$ in $\mathcal{S}'$ then $DT_n \to DT$ in $\mathcal{S}'$. In other words, the derivative in the distributional sense can always be exchanged with the weak limit. A consequence of this fact is that, if u_n is a regularization for T and u_n are differentiable functions, then u'_n is a regularization for T'.
2. For every distribution T, one can find a continuous function u which satisfies condition (3.1), such that $T = D^k u$ in the distributional sense for some integer k.

8 Fourth example: derivatives of delta function

The derivatives of delta functions have the following simple action on test functions

$$\langle \delta^{(k)}_{x_0}, \phi \rangle = (-1)^k \langle \delta_{x_0}, \phi^{(k)} \rangle = (-1)^k \phi^{(k)}(x_0) \ . \tag{8.1}$$

9 Product of distributions

In general the product of two distribution is not defined. Let us see why this is the case with an example. For instance, one may want to give meaning to the expression $\delta(x)\theta(x)$. A possible strategy is to consider a regularization u_n for the delta function. In this case $u_n(x)\theta(x)$ makes sense as a function. One may want to define $\delta(x)\theta(x)$ as the weak limit of $u_n(x)\theta(x)$ for $n \to \infty$. This definition is meaningful only if the limit does not depend on the chosen regularization. Now, we can see explicitly that this is not the case. For instance, if we choose the following regularizations (in the $\epsilon \to 0^+$ limit) for the delta function

$$\frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(x) \ , \qquad \frac{1}{\epsilon}\chi_{[0,\epsilon]}(x) \ , \qquad \frac{1}{\epsilon}\chi_{[-\epsilon,0]}(x) \ , \tag{9.1}$$

one gets different weak limits:

$$\lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(x)\theta(x) = \frac{1}{2}\delta(x) \ , \tag{9.2}$$

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\chi_{[0,\epsilon]}(x)\theta(x) = \delta(x) \ , \tag{9.3}$$

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\chi_{[-\epsilon,0]}(x)\theta(x) = 0 \ . \tag{9.4}$$

These results can be easily proven as follows

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \left\langle \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}\theta, \phi \right\rangle &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dx \ \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(x)\theta(x)\phi(x) \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon} \int_0^{\epsilon} dx \ \phi(x) = \frac{1}{2}\phi(0) = \frac{1}{2}\langle \delta_0, \phi \rangle \ , \qquad (9.5)\\
\lim_{\epsilon\to 0^+} \left\langle \frac{1}{\epsilon}\chi_{[0,\epsilon]}\theta, \phi \right\rangle &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dx \ \frac{1}{\epsilon}\chi_{[0,\epsilon]}(x)\theta(x)\phi(x) \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon} \int_0^{\epsilon} dx \ \phi(x) = \phi(0) = \langle \delta_0, \phi \rangle \ , \qquad (9.6)\\
\lim_{\epsilon\to 0^+} \left\langle \frac{1}{\epsilon}\chi_{[-\epsilon,0]}\theta, \phi \right\rangle &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dx \ \frac{1}{\epsilon}\chi_{[-\epsilon,0]}(x)\theta(x)\phi(x) = 0 \ , \qquad (9.7)
\end{aligned}$$

for a general test function ϕ.

The **product** of two distributions S and T can be meaningfully defined via regularization, only if it does not depend on the regularization. We will say that the product ST exists, if however we choose a regularization u_n for S and a regularization v_n for T, the sequence $u_n v_n$ converges in $\mathcal{S}'$, and the limit does not depend on the chosen regularizations. The following results hold (which we will give without proof):

1. For any distribution T and Schwartz function f, the product fT exists in $\mathcal{S}'$ and satisfies

   $$\langle fT, \phi \rangle = \langle T, f\phi \rangle \ . \tag{9.8}$$

2. Given a smooth function f with the property that f and all its derivatives are either bounded or divergent at most as polynomials at infinity, for any distribution T the product fT exists in $\mathcal{S}'$ and satisfies

   $$\langle fT, \phi \rangle = \langle T, f\phi \rangle \ . \tag{9.9}$$

3. Given a function f with the only property that f is continuous in x_0, then the product $f\delta_{x_0}$ exists in $\mathcal{S}'$ and

$$\langle f\delta_{x_0}, \phi\rangle = f(x_0)\phi(x_0) \ . \tag{9.10}$$

4. Given a function f with the property that f is k times differentiable in a neighbourhood of x_0, and its k-th derivative is continuous in x_0, then the product $f\delta^{(k)}_{x_0}$ exists in $\mathcal{S}'$ and satisfies

$$\langle f\delta^{(k)}_{x_0}, \phi\rangle = -\langle f'\delta^{(k-1)}, \phi\rangle - \langle f\delta^{(k-1)}_{x_0}, \phi'\rangle \ . \tag{9.11}$$

This formula allow to calculate $f\delta^{(k)}_{x_0}$ recursively.

5. If f is differentiable in zero, then the product $f(x)\mathrm{P}/x$ exists in $\mathcal{S}'$, and satisfies

$$\langle f(x)\frac{\mathrm{P}}{x}, \phi\rangle = \int_0^\infty dx\, \frac{f(x)\phi(x) - f(-x)\phi(-x)}{x} \ . \tag{9.12}$$

Compare this with eq. (5.3).

10 Selected problems

Problem 1. Calculate $\langle T, \phi\rangle$ as explicitly as possible, for any choice of the distribution

$$T = \delta_{x_0} \ , \quad \delta'_{x_0} \ , \quad \frac{\mathrm{P}}{x} \ , \quad x^2 \ , \tag{10.1}$$

and of the test function

$$\phi(x) = e^{-x^2} \ , \quad xe^{-x^2} \ , \quad e^{-x^2+ix} \ . \tag{10.2}$$

All these integrals can be calculated in terms of elementary functions, except one.

Problem 2. Prove that

$$\frac{x}{x^2+\epsilon^2} \tag{10.3}$$

is a regularization of P/x (in the $\epsilon \to 0^+$ limit).

Problem 3. Calculate the first, second and third derivative (in the sense of distributions) of $|x|$.

Problem 4. Let $f(x)$ be a continuous and differentiable function for every x except in $x = 0$, where it has a jump discontinuity. Let us assume also that f and f' (where it exists) are bounded. Show that f defines a regular distribution. Express the derivative of f in the sense of distribution in terms of the derivative of f in the sense of functions and of the jump at the discontinuity.

Problem 5. Prove that $\log|x|$ is a regular distribution, and

$$D\log|x| = \frac{\mathrm{P}}{x} \ , \tag{10.4}$$

where the derivative is taken in the sense of distributions.

Problem 6. Find an explicit expression for

$$\langle D\frac{\mathrm{P}}{x}, \phi\rangle \tag{10.5}$$

which does not involve limits.

Problem 7. Prove that

$$\lim_{\epsilon\to 0^+} \frac{1}{x+i\epsilon} = \frac{\mathrm{P}}{x} - i\pi\delta(x) \ , \tag{10.6}$$

where the limit has to be understood in the weak sense, i.e. in $\mathcal{S}'$. What is the following limit?

$$\lim_{\epsilon\to 0^+} \frac{1}{x-i\epsilon} \ . \tag{10.7}$$

Problem 8. Given some fixed x_0, show that every Schwartz function ϕ can be written as

$$\phi(x) = ke^{-(x-x_0)^2} + (x-x_0)\phi_1(x) \ , \tag{10.8}$$

where k is a constant and ϕ_1 is a Schwartz function (both k and ϕ_1 depend on ϕ). Find an explicit formula for k. Use this fact to show that, if T is a distribution that satisfies $(x-x_0)T = 0$, then a constant α exists such that $T = \alpha\delta_{x_0}$.

Problem 9. Generalize the argument of the previous problem to show that, if $(x-x_0)^2 T = 0$, then two constant α_0, α_1 exist such that $T = \alpha_0\delta_{x_0} + \alpha_1\delta'_{x_0}$.

Problem 10. Show that every Schwartz function ϕ can be written as

$$\phi(x) = ke^{-x^2} + \phi_1'(x) \ , \tag{10.9}$$

where k is a constant and ϕ_1 is a Schwartz function (both k and ϕ_1 depend on ϕ). Find an explicit formula for k. Use this fact to show that, if T is a distribution that satisfies $T' = 0$, then a constant α exists such that $T = \alpha$.

Problem 11. Show that $x \times \mathrm{P}/x = 1$ (where the product has to be interpreted as multiplication of distributions).

Problem 12. Use the results of problems 8 and 11 to write the most general distribution T that satisfies $xT = 1$.

Problem 13. Find the most general solution to the equation

$$T' = \delta_{x_0} \tag{10.10}$$

in distribution space. For a rigorous proof, you will need to use the result of problem 10.

Problem 14. Let f be a function with continuous derivative, which diverges for $x \to \pm\infty$, and with the property that $\inf_x f'(x) > 0$. Show that, for any test function ϕ, the following limit

$$\lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} dx \frac{e^{-\frac{[f(x)]^2}{2\sigma}}}{\sqrt{2\pi\sigma}} \phi(x) \tag{10.11}$$

exists finite, and defines a distribution which is usually denoted by $\delta(f(x))$. Write an explicit expression for this distribution. Some of the assumptions can be relaxed, which ones?