Lecture 1: Quantum free scalar field theory

1.1. Starting from the commutation relations for the ladder operators, show that $[\phi(x), \phi(y)] = 0$ if $(x - y)$ is space-like. This is the so-called *microcausality* property of the free scalar field.

Solution: Starting with the definition of the quantum scalar field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 E(p)} \left[a(p)e^{-iE(p)x_0+ipx} + a(p)^\dagger e^{-iE(p)x_0-ipx}\right] \tag{1}$$

and the commutation relations,

$$[a(p), a(q)^\dagger] = 2E(\boldsymbol{p})(2\pi)^3\delta^{(3)}(p-q), \tag{2}$$

$$[a(p), a(q)] = [a(p)^\dagger, a(q)^\dagger] = 0, \tag{3}$$

we can write the commutator of the scalar field as

$$\begin{aligned}
[\phi(x), \phi(y)] = &\int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})} \times \\
&\Big\{\Big[\cancel{a(p)e^{-iE(p)x_0+ipx}a(q)e^{-iE(q)y_0+iqy}} + a(p)e^{-iE(p)x_0+ipx}a(q)^\dagger e^{iE(q)y_0-iqy} \\
&+ a(p)^\dagger e^{iE(p)x_0-ipx}a(q)e^{-iE(q)y_0+iqy} + \cancel{a(p)^\dagger e^{iE(p)x_0-ipx}a(q)^\dagger e^{iE(q)y_0-iqy}}\Big] \\
&- \Big[\cancel{a(\boldsymbol{q})e^{-iE(\boldsymbol{q})y_0+i\boldsymbol{qy}}a(p)e^{-iE(p)x_0+ipx}} + a(q)e^{-iE(q)y_0+iqy}a(p)^\dagger e^{iE(p)y_0-ipx} \\
&+ a(q)^\dagger e^{iE(q)y_0-iqy}a(p)e^{-iE(p)x_0+ipx} + \cancel{a(q)^\dagger e^{iE(q)y_0-iqy}a(p)^\dagger e^{iE(p)x_0-ipx}}\Big]\Big\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
= &\int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})} \times \\
&\Big\{[a(p), a(q)^\dagger]e^{-iE(p)x_0+ipx}e^{iE(q)y_0-iqy} - [a(q), a(p)^\dagger]e^{iE(p)x_0-ipx}e^{-iE(q)y_0+iqy}\Big\}
\end{aligned} \tag{5}$$

$$\begin{aligned}
= &\int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}(2\pi)^3 2E(\boldsymbol{p})\delta^{(3)}(p-q) \times \\
&\Big\{e^{-iE(\boldsymbol{p})x_0+i\boldsymbol{px}}e^{iE(\boldsymbol{q})y_0-i\boldsymbol{qy}} - e^{iE(\boldsymbol{p})x_0-i\boldsymbol{px}}e^{-iE(\boldsymbol{q})y_0+i\boldsymbol{qy}}\Big\}
\end{aligned} \tag{6}$$

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \tag{7}$$

This is a Lorentz-invariant object, so to check if it vanishes for all space-like separations, we can evaluate it at one space-like vector; all are equivalent. Choose a frame where $x^0 = y^0 \implies x - y = z = (0, \boldsymbol{z})$. In this frame, you can see that the integral will vanish.

1.2. Prove that the components of the four-momentum operator commute with each other.

Solution: The four-momentum operator is

$$P^\mu = \int d^3x\, T^{0\mu}(x_0, \boldsymbol{x}), \tag{8}$$

where $T^{\mu\nu}$ is the energy-momentum tensor,

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2\right) - \epsilon_0 g^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}(x) - \epsilon_0 g^{\mu\nu}, \tag{9}$$

for some arbitary constant ϵ_0 and the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2. \tag{10}$$

We want to show that

$$[P^\mu, P^\nu] = 0. \tag{11}$$

First, recall the canonical momentum

$$\pi(x) = \partial_0\phi(x) \tag{12}$$

and its equal-time commutation relations with ϕ

$$[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{13}$$
$$[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0. \tag{14}$$

Now we expand the commutation of the four-momentum operator:

$$[P^\mu, P^\nu] = \left[\int d^3x\, T^{0\mu}(x_0,\boldsymbol{x}), \int d^3y\, T^{0\nu}(y_0,\boldsymbol{y})\right] \tag{15}$$

$$\begin{aligned} = \int d^3x\, d^3y \Bigg\{ \Bigg[&\left(\pi(x)\partial^\mu\phi(x) - g^{0\mu}\left(\frac{1}{2}\partial_\rho\phi(x)\partial^\rho\phi(x) - \frac{1}{2}m^2\phi^2(x)\right) + \cancel{\epsilon_0 g^{0\mu}}\right) \\ &\times \left(\pi(y)\partial^\nu\phi(y) - g^{0\nu}\left(\frac{1}{2}\partial_\lambda\phi(y)\partial^\lambda\phi(y) - \frac{1}{2}m^2\phi^2(y)\right) + \cancel{\epsilon_0 g^{0\nu}}\right)\Bigg] \\ - \Bigg[&\left(\pi(y)\partial^\nu\phi(y) - g^{0\nu}\left(\frac{1}{2}\partial_\lambda\phi(y)\partial^\lambda\phi(y) - \frac{1}{2}m^2\phi^2(y)\right) + \cancel{\epsilon_0 g^{0\nu}}\right) \\ &\times \left(\pi(x)\partial^\mu\phi(x) - g^{0\mu}\left(\frac{1}{2}\partial_\rho\phi(x)\partial^\rho\phi(x) - \frac{1}{2}m^2\phi^2(x)\right) + \cancel{\epsilon_0 g^{0\mu}}\right)\Bigg]\Bigg\} \end{aligned} \tag{16}$$

where the ϵ_0 term cancels trivially.

$$\begin{aligned} [P^\mu, P^\nu] = \int d^3x\, d^3y \Big\{ &[\pi(x)\partial^\mu\phi(x), \pi(y)\partial^\nu\phi(y)] + \cancel{[g^{0\mu}\mathcal{L}(x), g^{0\nu}\mathcal{L}(y)]} \\ &+ [\pi(x)\partial^\mu\phi(x), g^{0\nu}\mathcal{L}(y)] + [g^{0\mu}\mathcal{L}(x), \pi(y)\partial^\nu\phi(y)]\Big\} \end{aligned} \tag{17}$$

where we have now compressed the energy-momentum tensor contribution of the Lagrangian density, and since the Lagrangian density trivially commutes with itself we have already cancelled this term. Let's look at separate cases of μ, ν:

- Case 1, $\mu, \nu \to i, j \neq 0$:

$$[P^i, P^j] = \int d^3x\, d^3y\, [\pi(x)\partial^i\phi(x), \pi(y)\partial^j\phi(y)] \tag{18}$$

since the metric tensor will remove the Lagrangian in this case. Then we can use the Leibniz rule,

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B, \tag{19}$$

to expand as

$$[P^i, P^j] = \int d^3x\, d^3y \Big\{ \pi(x)[\partial^i\phi(x), \pi(y)]\partial^j(\phi(y) + \pi(x)\pi(y)\cancel{[\partial^i\phi(x), \partial^j\phi(y)]} + \cancel{[\pi(x), \pi(y)]}\partial^i\phi(x)\partial^j\phi(y) + \pi(y)[\pi(x), \partial^j\phi(y)]\partial^j\phi(x) \Big\} \tag{20}$$

where we see that two commutators are simply zero. The remaining commutators are found using the identity

$$[\partial^i\phi(x), \pi(y)] = i\partial^i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \tag{21}$$

leading to

$$[P^i, P^j] = \int d^3x\, d^3y \Big\{ i\pi(x)\partial^i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\partial^j\phi(y) - i\pi(y)\partial^j\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\partial^i\phi(x) \Big\}. \tag{22}$$

The integral of a derivative of a delta function with some test function is given by

$$\int d^3y\, \partial^i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})f(\boldsymbol{y}) = -\partial^i f(\boldsymbol{x}) \tag{23}$$

and so the commutator becomes

$$[P^i, P^j] = i\int d^3x\, \cancel{\Big\{ -\pi(x)\partial^i\partial^j\phi(x) + \pi(x)\partial^j\partial^i\phi(x) \Big\}} = 0. \tag{24}$$

➤ Case 2, $\mu = 0, \nu = i$: When $\mu = 0$, P^0 becomes the Hamiltonian:

$$P^0 = \int d^3x \Big\{ \partial^0\phi\partial^0\phi - \Big(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2\Big) - \epsilon_0 \Big\} \tag{25}$$

$$= \int d^3x \Big\{ \partial^0\phi\partial^0\phi - \frac{1}{2}\partial^0\phi\partial^0\phi + \frac{1}{2}\partial^i\phi\partial^j\phi + \frac{1}{2}m^2\phi^2 - \epsilon_0 \Big\} \tag{26}$$

$$= \int d^3x \Big\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \epsilon_0 \Big\} = \hat{\mathcal{H}} \tag{27}$$

So then the commutator is

$$[\hat{\mathcal{H}}, P^i] = \Big[\int d^3x \Big\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \epsilon_0 \Big\}, \int d^3y\, \pi(y)\partial^j\phi(y) \Big] \tag{28}$$

Let's take this term by term:

➥ Term 1:

$$\Big[\frac{1}{2}\pi^2(x), \pi(y)\partial^i\phi(y) \Big] = \frac{1}{2}\pi(x)\cancel{[\pi(x), \pi(y)]}\partial^i\phi(y) + \frac{1}{2}\pi(x)\pi(y)[\pi(x), \partial^i\phi(y)] + \frac{1}{2}\cancel{[\pi(x), \pi(y)]}\partial^i\phi(y)\pi(x) + \frac{1}{2}\pi(y)[\pi(x), \partial^i\phi(y)]\pi(x) \tag{29}$$

Now perform the integral:

$$\begin{aligned}
&\frac{1}{2}\int d^3x\, d^3y \left\{\pi(x)\pi(y)[\pi(x), \partial^i\phi(y)] + \pi(y)[\pi(x), \partial^i\phi(y)]\pi(x)\right\} \\
&= -\frac{i}{2}\int d^3x\, d^3y\ [\pi(x)\pi(y) + \pi(y)\pi(x)]\,\partial^i\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (30)\\
&= \frac{i}{2}\int d^3x\,\partial^i(\pi(x)^2) = 0. \qquad (31)
\end{aligned}$$

➥ Term 2:

$$\left[\frac{1}{2}(\nabla\phi(x))^2, \pi(y)\partial^i\phi(y)\right] = \frac{1}{2}\left[\partial^j\phi(x)\partial^j\phi(x), \pi(x)\partial^i\phi(y)\right] \qquad (32)$$

$$\begin{aligned}
&= \frac{1}{2}\Big\{\partial^j\phi(x)\left[\partial^j\phi(x), \pi(y)\right]\partial^i\phi(x) + \partial^j\phi(x)\pi(y)\cancel{[\partial^j\phi(x), \partial^i\phi(y)]} \\
&\quad + \left[\partial^j\phi(x), \pi(y)\right]\partial^i\phi(y)\partial^j\phi(x) + \pi(y)\cancel{[\partial^j\phi(x), \partial^i\phi(y)]}\partial^j\phi(x)\Big\} \qquad (33)\\
&= i\nabla\phi(x)\cdot\nabla\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\partial^i\phi(y) \qquad (34)
\end{aligned}$$

Now integrating this result,

$$\int d^3x\, d^3y\, i\partial^j\phi(x)\cdot\partial^j\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\partial^i\phi(y) = -i\int d^3x\,\partial^j\left(\partial^j\phi(x)\partial^i\phi(x)\right) = 0 \qquad (35)$$

➥ Term 3:

$$\begin{aligned}
&\left[\frac{1}{2}m^2\phi^2(x), \pi(y)\partial^i\phi(y)\right] \\
&= \Big\{\frac{1}{2}m^2\phi(x)\left[\phi(x), \pi(y)\right]\partial^i\phi(y) + \frac{1}{2}m^2\phi(x)\pi(y)\cancel{[\phi(x), \partial^i\phi(y)]} \\
&\quad + \frac{1}{2}m^2\left[\phi(x), \pi(y)\right]\partial^i\phi(y)(\phi(x) + \frac{1}{2}m^2\pi(y)\cancel{[\phi(x), \partial^i\phi(y)]}\phi(x)\Big\} \qquad (36)\\
&= im^2\phi(x)\partial^i\phi(y)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad (37)
\end{aligned}$$

Now integrating,

$$\int d^3x\, d^3y\, im^2\phi(x)\partial^i\phi(y)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = im^2\int d^3x\,\phi(x)\partial^i\phi(x) = 0 \qquad (38)$$

➥ Term 4 trivally commutes

1.3. Prove that the operators $\mathbb{P}_n$ are mutually orthogonal projectors on the n-particle sector of the Fock space.

Solution: We want to show that

$$\mathbb{P}_n^2 = \mathbb{P}_n, \qquad (39)$$

$$\mathbb{P}_n^\dagger = \mathbb{P}_n, \qquad (40)$$

$$\mathbb{P}_n\mathbb{P}_m = 0 \text{ for } n \neq m. \qquad (41)$$

- Let $|\psi\rangle \in \mathcal{F}$, with

$$\mathbb{P}_n^2 = \mathbb{P}_n\mathbb{P}_n = \left(\frac{1}{n!}\int d\Pi_n\,|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n|\right)^2 \tag{42}$$

$$= \frac{1}{(n!)^2}\int d\Pi_n\,d\Pi_n'|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n|p_1',\ldots,p_n'\rangle\langle p_1',\ldots,p_n'|. \tag{43}$$

Using the orthonormality of the n-particle states, the inner product gives us

$$\langle p_1,\ldots,p_n|p_1',\ldots,p_n'\rangle = n!\prod_{k=1}^{n}(2\pi)^3\,2E(\boldsymbol{p}_k)\delta^{(3)}(\boldsymbol{p}_k-\boldsymbol{p}_k'). \tag{44}$$

Then,

$$\mathbb{P}_n^2 = \frac{1}{(n!)^2}\int d\Pi_n\,d\Pi_n'\,|p_1,\ldots,p_n\rangle\left[n!\prod_{k=1}^{n}(2\pi)^3\,2E(\boldsymbol{p}_k)\,\delta^{(3)}(\boldsymbol{p}_k-\boldsymbol{p}_k')\right]\langle p_1',\ldots,p_n| \tag{45}$$

$$= \frac{1}{n!}\int d\Pi_n\,|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n| = \mathbb{P}_n. \tag{46}$$

- Take the Hermitian conjugate of the projector:

$$\mathbb{P}_n^\dagger = \left(\frac{1}{n!}\int d\Pi_n\,|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n|\right)^\dagger = \frac{1}{n!}\int d\Pi_n\,|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n| = \mathbb{P}_n. \tag{47}$$

- Knowing that the n-particle momentum eigenstates $|p_1,\ldots,p_n\rangle$ are normalised, then

$$\mathbb{P}_n\mathbb{P}_m = \left[\frac{1}{n!}\int d\Pi_n|p_1\ldots,p_n\rangle\langle p_1,\ldots,p_n|\right]\left[\frac{1}{m!}\int d\Pi_m|q_1\ldots,q_n\rangle\langle q_1,\ldots,q_n|\right] \tag{48}$$

$$= \frac{1}{n!\,m!}\int d\Pi_n\,d\Pi_m\,|p_1,\ldots,p_n\rangle\langle p_1,\ldots,p_n|q_1,\ldots,q_m\rangle\langle q_1,\ldots,q_m| \tag{49}$$

Since the Fock space is an orthogonal sum,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty}\mathcal{H}^{(n)}, \tag{50}$$

the Hilbert spaces $\mathcal{H}^{(k)}$ of k-particle states are orthogonal to one another. Therefore the inner product in $\mathbb{P}_n\mathbb{P}_m$ must be 0:

$$\langle p_1,\ldots,p_n|q_1,\ldots,q_m\rangle = 0, \tag{51}$$

which therefore makes the whole expression 0.

1.4. Prove that the state $|p_1,p_2,\ldots,p_n\rangle$ are eigenstates of the four-momentum and calculate their eigenvalues.

Solution: An n-particle state is

$$|p_1,\ldots,p_n\rangle = a^\dagger(p_1)\cdots a^\dagger(p_n)|\Omega\rangle \tag{52}$$

and the four-momentum operator can be written

$$P = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} p\, a^\dagger(p) a(p). \tag{53}$$

It is trivial to show that

$$[P^\mu, a^\dagger(p)] = p^\mu a^\dagger(p), \tag{54}$$

and then we can see that acting on a single creation operator can be written as

$$P^\mu a^\dagger(p_1) = a^\dagger(p_1) P^\mu + p_1^\mu a^\dagger(p_1). \tag{55}$$

This relation can be applied recursively to a n-particle state, yielding

$$P^\mu |p_1, \ldots, p_n\rangle = P^\mu a^\dagger(p_1) \cdots a^\dagger(p_n) |\Omega\rangle \tag{56}$$

$$= \left(\sum_{i=1}^n p_i^\mu \right) |p_1, \ldots, p_n\rangle, \tag{57}$$

since $P^\mu |0\rangle = 0$.

1.5. Prove that

$$\{p_1 + p_2 \text{ s.t. } p_1,\, p_2 \in \mathcal{M}_1\} = \left\{ p \in \mathbb{R}^4 \text{ s.t. } p^2 \geq (2m)^2 \text{ and } p_0 > 0 \right\}. \tag{58}$$

Solution: Reminder:

$$\mathcal{M}_1 = \left\{ p \in \mathbb{R}^4 \text{ s.t. } p^2 = m^2 \text{ and } p_0 > 0 \right\}. \tag{59}$$

Let $p \in \mathbb{R}^4$ with $p^2 \geq (2m)^2$ and $p_0 > 0$. We need to find $p_1, p_2 \in \mathcal{M}_1$ such that $p = p_1 + p_2$. Choose the rest frame of p: define a Lorentz frame where $p = (p_0, \boldsymbol{0})$, and then $p_1 = (\frac{p_0}{2}, \boldsymbol{q})$ and $p_2 = (\frac{p_0}{2}, -\boldsymbol{q}$ such that $p = p_1 + p_2$. Then

$$p^2 = (p_1 + p_2)^2 = \left\{ \left(\sqrt{m^2 + \boldsymbol{p}_1^2}, \boldsymbol{0} \right) + \left(\sqrt{m^2 + \boldsymbol{p}_2^2}, \boldsymbol{0} \right) \right\}^2 \tag{60}$$

$$= \left(2\sqrt{m^2 + q^2}, \boldsymbol{0} \right)^2 = (2m)^2 + 4q^2, \quad q^2 > 0. \tag{61}$$

1.6. Prove that the following operator

$$L^{\mu\nu} = \int d^3x \left[x^\nu T^{0\mu}(x) - x^\mu T^{0\nu}(x) \right] \tag{62}$$

does not depend on x^0 and that it generates Lorentz transformations on the field $\phi(x)$. In particular, for any real matrix $\omega_{\mu\nu}$ satisfying $\omega_{\mu\nu} - \omega_{\nu\mu}$, show that:

$$e^{-\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}} \phi(x) e^{\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}} = \phi(\Lambda^{-1} x) \tag{63}$$

where $\Lambda = e^\omega$ is a Lorentz transformation.

Solution: To show that $L^{\mu\nu}$ does not depend on x^0, we want

$$\frac{d}{dx^0} L^{\mu\nu}(x^0) = 0 \tag{64}$$

$$= \int d^3x \, \left[x^\nu \partial_0 T^{0\mu}(x) - x^\mu \partial_0 T^{0\nu}(x) \right]. \tag{65}$$

The energy-momentum tensor satisfies local conservation $\partial_\lambda T^{\lambda\mu} = 0$, in particular

$$\partial_0 T^{0\mu} = -\partial_i T^{i\mu}, \tag{66}$$

and thus

$$\frac{d}{dx^0} L^{\mu\nu}(x^0) = -\int d^3x \, \left[x^\nu \partial_i T^{i\mu} - x^\mu \partial_i T^{i\nu} \right]. \tag{67}$$

Now we perform IBP on each term:

$$\int d^3x \, x^\nu \partial_i T^{i\mu} = \int d^3x \, \partial_i (x^\nu T^{i\mu}) - \int d^3x \, \partial_i^\nu T^{i\mu} = -\int d^3x \, T^{\nu\mu}, \tag{68}$$

where we can set the surface term to 0. Substituting back in, we find

$$\frac{d}{dx^0} L^{\mu\nu}(x^0) = \int d^3x \, T^{\nu\mu} - \int d^3x \, T^{\mu\nu} = 0. \tag{69}$$

Since the energy-momentum tensor is symmetric, i.e. $T^{\mu\nu} = T^{\nu\mu}$, the two terms cancel.
To show that $L^{\mu\nu}$ generates Lorentz transformations of a field $\phi(x)$, we aim to write

$$\delta\phi(x) = \frac{i}{2} \omega_{\mu\nu} [L^{\mu\nu}, \phi(x)] = -\omega^{\mu\nu} x_\nu \partial_\mu \phi(x), \tag{70}$$

which is the infinitesimal Lorentz transformation of $\phi(x)$.

- An infinitesimal Lorentz transformation acts on spacetime coordinates as

$$x^\mu \to x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \tag{71}$$

  where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the infinitesimal antisymmetric paramters of the Lorentz group. Under this transformation, a scalar field $\phi(x)$ transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \approx \phi(x - \omega x) = \phi(x) - \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \tag{72}$$

  So the variation in ϕ is

$$\delta\phi(x) = \phi'(x) - \phi(x) = -\omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \tag{73}$$

- We can write the commutator identity

$$\left[T^{0\mu}(y), \phi(x) \right] = -i \partial^\mu \phi(x) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{74}$$

  which tells us that $T^{0\mu}(y)$ generates translations in the x^μ direction.

- Now we want to commute the commutator of $L^{\mu\nu}$ with ϕ:

$$[L^{\mu\nu}(y), \phi(x)] = \int d^3y \left\{ x^\nu \left[T^{0\mu}(y), \phi(x) \right] - x^\mu \left[T^{0\nu}(y), \phi(x) \right] \right\} \tag{75}$$

$$= -i \int d^3y \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \, \left[x^\nu \partial^\mu \phi(x) - x^\mu \partial^\nu \phi(x) \right] \tag{76}$$

$$= -i \, (x^\nu \partial^\mu) - x^\mu \partial^\nu) \, \phi(x \tag{77}$$

- Now we can recover the transformation:

$$\delta\phi(x) = \frac{i}{2}\omega_{\mu\nu}\left[L^{\mu\nu}, \phi(x)\right] = \frac{i}{2}\omega_{\mu\nu}\left[-i\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\phi(x)\right] \tag{78}$$

$$= \frac{1}{2}\omega_{\mu\nu}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\phi(x) = -\omega^{\mu\nu}x_\nu\partial_\mu\phi(x) \tag{79}$$

We want to calculate

$$e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}\phi(x)e^{\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}. \tag{80}$$

We can use the BCH identity

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots, \tag{81}$$

where in our case $A = -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}$. So

$$e^A\phi(x)e^{-A} = \phi(x) + [A, \phi(x)] + \frac{1}{2!}[A, [A, \phi(x)]] + \dots, \tag{82}$$

but this is just the Taylor expansion of $\phi(\Lambda^{-1}x)$ since

$$\delta\phi(x) = -\omega^{\mu\nu}x_\nu\partial_\mu\phi(x) \tag{83}$$

is the infinitesimal shift under $x \to x' = \Lambda^{-1}x$.