Lecture 2: Axiomatic framework and QCD

2.1. Prove that the smeared fields transform under translation in the same way as local fields, i.e.

$$\phi^f(x+a) = e^{iPa}\phi^f(x)e^{-iPa}. \tag{1}$$

What happens under Lorentz transformation?

Solution: The smeared field at x is defined

$$\phi^f(x) = \int d^4y\, f(x-y)\phi(y). \tag{2}$$

So the smeared field at $x+a$ can be written

$$\phi^f(x+a) = \int d^4y\, f(x+a-y)\phi(y) \tag{3}$$

$$y' = y - a \implies = \int d^4y'\, f(x-y')\phi(y'+a). \tag{4}$$

Now consider the translation:

$$e^{iPa}\phi^f(x)e^{-iPa} = e^{iPa}\left(\int d^4y\, f(x-y)\phi(y)\right)e^{-iPa} \tag{5}$$

$$= \int d^4y\, f(x-y)e^{iPa}\phi(y)e^{-iPa} \tag{6}$$

$$= \int d^4y\, f(x-y)\phi(y+a) \tag{7}$$

$$= \phi^f(x+a). \tag{8}$$

Recalling that the Lorentz translation of the regular field is

$$U(\Lambda,a)\phi(x)U^{-1}(\Lambda,a) = R(\Lambda^{-1})\phi(\Lambda x + a), \tag{9}$$

we consider the Lorentz transformation of the smeared field:

$$U(\Lambda,a)\phi^f(x)U^{-1}(\Lambda,a) = U(\Lambda,a)\left(\int d^4y\, f(x-y)\phi(y)\right)U^{-1}(\Lambda,a) \tag{10}$$

$$= \int d^4y\, f(x-y)U(\Lambda,a)\phi(y)U^{-1}(\Lambda,a) \tag{11}$$

$$= \int d^4y\, f(x-y)R(\Lambda^{-1})\phi(\Lambda y + a) \tag{12}$$

$$= R(\Lambda^{-1})\int d^4y\, f(x-y)\phi(\Lambda y + a) \tag{13}$$

$$y' = \Lambda y \implies = R(\Lambda^{-1})\int d^4y'\, f(x-\Lambda^{-1}y')\phi(y') \tag{14}$$

$$= R(\Lambda^{-1})\int d^4y'\, f_\Lambda(\Lambda x - y')\phi(y') \tag{15}$$

$$= R(\Lambda^{-1})\phi^{f_\Lambda}(\Lambda x), \tag{16}$$

where we have transformed the test function $f \to f_\Lambda(x) = f(\Lambda^{-1}x)$. The smeared field transforms like a field at Λx with a transformed smearing profile.

2.2. Define the Fourier transform of a field operator $\phi(x)$ as

$$\tilde{\phi}(p) = \int d^4x\, e^{ipx}\phi(x). \tag{17}$$

Let $|p\rangle$ be an eigenstate of the four-momentum operator P_μ with eigenvalue p_μ. Prove that $\tilde{\phi}(q)|p\rangle$ and $\tilde{\phi}^\dagger(q)|p\rangle$ are eigenstates of P_μ and calculate the corresponding eigenvalues. Argue that $\tilde{\phi}^\dagger(q)$ and $\tilde{\phi}(q)$ act as creation and annihilation operators for the four momentum.

Solution: The momentum operator generates translations as

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x), \tag{18}$$

which implies a commutator also exists for the Fourier-transformed field:

$$[P^\mu, \tilde{\phi}(q)] = \int d^4x\, e^{iqx}[P^\mu, \phi(x)] = -i\int d^4x\, e^{iqx}\partial^\mu\phi(x). \tag{19}$$

This can be evaluated using IBP:

$$[P^\mu, \tilde{\phi}(q)] = -i\left(-\int d^4x\,(\partial^\mu e^{iqx})\phi(x)\right) \tag{20}$$

$$= i\int d^4x iq^\mu e^{iqx}\phi(x) \tag{21}$$

$$= -q^\mu\tilde{\phi}(q). \tag{22}$$

So then we can consider

$$P^\mu\left(\tilde{\phi}(q)|p\rangle\right) = \left(P^\mu\tilde{\phi}(q)\right)|p\rangle \tag{23}$$

$$= \left(\tilde{\phi}(q)P^\mu + [P^\mu, \tilde{\phi}(q)]\right)|p\rangle \tag{24}$$

$$= \tilde{\phi}(q)P^\mu|p\rangle + [P^\mu, \tilde{\phi}(q)]|p\rangle \tag{25}$$

$$= p^\mu\tilde{\phi}(q)|p\rangle - q^\mu\tilde{\phi}(q)|p\rangle \tag{26}$$

$$= (p^\mu - q^\mu)\,\tilde{\phi}(q)|p\rangle. \tag{27}$$

Similarly, with the commutator

$$[P^\mu, \phi^\dagger(x)] = i\partial_\mu\phi^\dagger(x) \implies [P^\mu, \tilde{\phi}^\dagger(q)] = q^\mu\tilde{\phi}^\dagger(q). \tag{28}$$

Then,

$$P^\mu\left(\tilde{\phi}^\dagger(q)|p\rangle\right) = (p^\mu + q^\mu)\tilde{\phi}^\dagger|p\rangle. \tag{29}$$

So $\tilde{\phi}^\dagger(q)$ and $\tilde{\phi}(q)$ act as creation and annihilation operators on the four-momentum by injecting or removing momentum from the state.

2.3. Consider two observables A and B which, for simplicity, are assumed to have only discrete non-degenerate eigenvalues. Denote by a_n and b_m the eigenvalues of A and B, respectively, and by $|a_n\rangle$ and $|b_m\rangle$ the corresponding eigenstates. Consider a generic normalised state $|\psi\rangle$. We imagine two different measurement protocols:

1. We measure A on the state $|\psi\rangle$ first, and then B;

2. We measure B on the state $|\psi\rangle$ first, and then A.

Recall that the state changes as a consequence of the measurement procedure. Let $p_{n,m}$ (resp. $q_{n,m}$) be the probability of obtaining a_n as the result of the measurement of A and b_m as the result of the measurement of B in the first (resp. second) protocol. Write a formula for $p_{n,m}$ and $q_{n,m}$ in terms of the eigenstates $|a_n\rangle$ and $|b_m\rangle$ and the state $|\psi\rangle$. Show that the order in which the observables are measured does not matter (for any state $|\psi\rangle$) if and only if the two observables commute.

Solution: Let's look at the two protocols separately:

1. First, measure A. The probability of outcome a_n is
$$P(a_n) = |\langle a_n|\psi\rangle|^2, \tag{30}$$
and the post-measurement state collapses to $|a_n\rangle$. Now, measure B on $|a_n\rangle$. The probability of outcome b_m is
$$P(b_m|a_n) = |\langle b_m|a_n\rangle|^2. \tag{31}$$
The total joint probability is then
$$p_{n,m} = P(a_n)P(b_m|a_n) = |\langle a_n|\psi\rangle|^2 \cdot |\langle b_m|a_n\rangle|^2. \tag{32}$$
2. First, measure B. The probability of outcome b_m is
$$P(b_m) = |\langle b_m|\psi\rangle|^2, \tag{33}$$
and the post-measurement state collapses to $|b_m\rangle$. Now, measure A on $|b_m\rangle$. The probability of outcome a_n is
$$P(a_n|b_m) = |\langle a_n|b_m\rangle|^2. \tag{34}$$
The total joint probability is then
$$q_{n,m} = P(b_m)P(a_n|b_m) = |\langle b_m|\psi\rangle|^2 \cdot |\langle a_n|b_m\rangle|^2. \tag{35}$$

Now let's show that these are equivalent if A and B commute. If $[A, B] = 0$, then there exists a common orthonormal basis $\{|c_k\rangle\}$ such that
$$A|c_k\rangle = a_k|c_k\rangle, \quad B|c_k\rangle = b_k|c_k\rangle \implies |a_n\rangle = |b_m\rangle \text{ if } a_n = a_m \text{ etc.} \tag{36}$$
Therefore, $\langle a_n|b_m\rangle = \delta_{nm}$ and then the probabilities are
$$p_{n,m} = |\langle a_n|\psi\rangle|^2 \cdot \delta_{n,m}, \tag{37}$$
$$q_{n,m} = |\langle b_n|\psi\rangle|^2 \cdot \delta_{n,m} = |\langle a_n|\psi\rangle|^2 \cdot \delta_{n,m}. \tag{38}$$
So $p_{n,m} = q_{n,m}$ if $[A, B] = 0$.

Now the **only if**: Suppose $p_{n,m} = q_{n,m}$ for all normalised states $|\psi\rangle$, then
$$|\langle a_n|\psi\rangle|^2 \cdot |\langle b_m|a_n\rangle|^2 = |\langle b_m|\psi\rangle|^2 \cdot |\langle a_n|b_m\rangle|^2. \tag{39}$$
Choose for example $|\psi\rangle = |a_k\rangle$, then find
$$\delta_{nk} \cdot |\langle b_m|a_n\rangle|^2 = |\langle b_m|a_k\rangle| \cdot |\langle a_n|b_m\rangle|^2. \tag{40}$$
If we assume $|\langle b_m|a_k\rangle|^2 \neq 0$, then we find
$$\delta_{nk} = |\langle a_n|b_m\rangle|^2. \tag{41}$$
Or otherwise,
$$|\langle a_n|b_m\rangle|^2 = \delta_{n,f(m)}, \tag{42}$$
for some unique mapping $f(m) \to n$.

2.4. In the free scalar case, prove the formulae:

$$\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot x}}{p^2-m^2+i\epsilon}, \tag{43}$$

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\theta(p_0)\,2\pi\,\delta(p^2-m^2)e^{-ip\cdot x}. \tag{44}$$

Solution:

- First, we will compute eq. (44). The scalar field operator has the expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right]. \tag{45}$$

  We insert this into eq. (44), finding

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\frac{d^3q}{(2\pi)^3 E(\boldsymbol{q})}\langle\Omega|\left[a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\right]\left[a(q)+a^\dagger(q)\right]|\Omega\rangle \tag{46}$$

  The only non-zero contribution comes from

$$\langle\Omega|a(p)a^\dagger(q)|\Omega\rangle = 2E(\boldsymbol{p})(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{47}$$

  leading to

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}e^{-ip\cdot x}. \tag{48}$$

  This is the standard result of the Wightman function in terms of a 3-momentum integral, but now we want to express this as a 4-momentum integral. We require:

  - ➡ a delta function $\delta(p^2-m^2)$ to enforce the mass-shell condition $p^0 = \pm\sqrt{p^2+m^2}$;
  - ➡ a $\theta(p_0)$ to pick out the positive frequency $p_0 > 0$;
  - ➡ the Jacobian of the delta function integral yields

$$\int d^4p\,\delta(p^2-m^2)\theta(p_0)f(p) = \int \frac{d^3p}{2E(\boldsymbol{p})}f(E(\boldsymbol{p}),p). \tag{49}$$

  So now we can write the 4-momentum integral as

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\theta(p_0)\,2\pi\,\delta(p^2-m^2)e^{-ip\cdot x}. \tag{50}$$

- Now for eq. (43), we want to compute the Feynman propagator in position space,

$$i\Delta_F(x) = \langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle. \tag{51}$$

  The time-ordered product $T\{\phi(x)\phi(0)\}$ is defined

$$T\{\phi(x)\phi(0)\} = \begin{cases}\phi(x)\phi(0), & \text{if } x^0 > 0,\\ \phi(0)\phi(x), & \text{if } x^0 < 0,\end{cases} \tag{52}$$

so the VEV of this is

$$\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle = \theta(x_0)\langle\Omega|\phi(x)\phi(0)|\Omega\rangle + \theta(-x_0)\langle\Omega|\phi(0)\phi(x)|\Omega\rangle. \tag{53}$$

We know from the 3-momentum Wightman function above that

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} e^{-ip\cdot x}, \tag{54}$$

$$\langle\Omega|\phi(0)\phi(x)|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} e^{ip\cdot x}, \tag{55}$$

and so the time-ordered expression is

$$\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \left[\theta(x_0)e^{-ip\cdot x} + \theta(-x_0)e^{ip\cdot x}\right]. \tag{56}$$

Again, we want to rewrite this as a 4-momentum integral. If we consider the p_0 integral of eq. (43) explicitly,

$$\int \frac{dp_0}{2\pi} \frac{e^{-ip_0x_0}}{(p_0)^2 - E(\boldsymbol{p})^2 + i\epsilon}. \tag{57}$$

This integral can be done via contour integration with poles at $p_0 = \pm E(\boldsymbol{p}) \mp i\epsilon$. By closing the contour:

- If $x^0 > 0$, we close in the lower half-plane and pick up $p^0 = E(\boldsymbol{p}) - i\epsilon$;
- If $x^0 < 0$, we close in the upper half-plane and pick up $p^0 = -E(\boldsymbol{p}) + i\epsilon$.

The residues give

$$\int \frac{dp_0}{2\pi} \frac{e^{-ip_0x_0}}{(p_0)^2 - E(\boldsymbol{p})^2 + i\epsilon} = \frac{1}{2E(\boldsymbol{p})} \left[\theta(x_0)e^{-iE(\boldsymbol{p})x_0} + \theta(-x_0)e^{iE(\boldsymbol{p})x_0}\right] \tag{58}$$

and then reincluding the 3-momentum integral, this will return us to eq. (56). Thus, eq. (56) is equivalent to eq. (43).

2.5. In operatorial formalism and temporal gauge ($A_0 = 0$), QCD can be described in terms of the following fundamental fields in the Schrödinger picture:

- the gluon field $A_k(x) = \sum_a A_k^a(x)T^a$, where $k = 1, 2, 3$ is the spatial index, $a = 1, \ldots, 8$ is the colour index, and T^a are the generators the gauge group SU(3 with the normalisation $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$; for instance, one can choose $T^a = \lambda^a/2$ where λ^a are the Gell-Mann matrices;
- the chromoelectric field $E_k(x) = \sum_a E_k^a(x)T^a$, with the same conventions as for the gluon field;
- the quark fields $\psi_{\alpha i f}(x)$, where $\alpha = 1, 2, 3, 4$ is the Dirac spinor index, $i = 1, 2, 3$ is the colour index and $f = u, d$ is the flavour index.

The fundamental fields satisfy canonical equal-time (anti-)commutation relations:

$$[A_j^a(x), E_k^b(y)] = i\delta_{jk}\delta^{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{59}$$

$$\{\psi_f(x), \psi_g^\dagger(y)\} = I_{12\times 12}\delta_{fg}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{60}$$

Consider the operator

$$G(x) = \sum_a G^a(x)\lambda^a, \tag{61}$$

$$G^a(x) = D_k E_k^a(x) + \sum_f \psi_f^\dagger(x) T^a \psi_f(x), \tag{62}$$

and its smeared version

$$G(\omega) = 2\int d^3x\, \mathrm{tr}[\omega(x)G(x)], \tag{63}$$

where $\omega(x) = \sum_a \omega^a(x)T^a$ is a test function which is smooth and vanish at infinity. Notice that we smear here only in space, not in time.
Calculate the commutators of $G(\omega)$ with the fundamental fields $A_k^a(x)$, $E_k^a(x)$, and $\psi_f(x)$. Argue that $G(x)$ can be interpreted as the generator of gauge transformations and that physical states must satisfy the condition $G(x)|\Psi\rangle = 0$.

Solution: First, we will rewrite $G(\omega)$ by propagating $\omega^a(x)$ through the expression for $G^a(x)$:

$$G(\omega) = \int d^3x \left\{ -E_k^a(x)(D_k\omega(x))^a + \sum_f \psi_f^\dagger \omega(x)\psi_f(x) \right\}, \tag{64}$$

where we have reordered the derivative on the gauge field part using IBP and vanishing boundary terms.

- Commutator with $A_j^b(y)$. Since $A_j^b(y)$ commutes with the fermion fields ψ, we are only concerned with the gauge field part of $G(\omega)$:

$$[G(\omega), A_j^b(y)] = -\int d^3x\, [E_k^a(x), A_j^b(y)]\, (D_k\omega(x))^a = iD_j\omega^b(y), \tag{65}$$

where we have used eq. (59). Note that this is exactly how the gauge field transform under an infinitesimal gauge transformation, $\delta_w A_j^b = D_j\omega^b$.

- Commutator with $E_j^b(y)$. Again, $E_j^b(y)$ commutes with the fermion fields, and also with itself, and so naively it would be 0. However, we should realise that $E_j^b(x)$ does not commute with $A_i^c(y)$ and so we should expand the covariant derivative, $(D_kE^k)^a = \partial_k E^{k,a} + f^{abc}A_i^b E^{i,c}$:

$$[G(\omega), E_j^d(y)] = \int d^3x\, \omega^a(x) g f^{abc}[A_i^b(x)E^{i,c}(x), E_j^d(y)] \tag{66}$$

$$= i\int d^3x\, \omega^a(x) f^{abc}\delta^{bd}\delta_{ij}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})E^{i,c}(x) \tag{67}$$

$$= i f^{abc}\omega^a(y)E^{j,c}(y). \tag{68}$$

This again represents an infinitesimal gauge transformation in the adjoint representation.

- Commutator with $\psi_f(y)$. First we can recognise that the quark field fully commutes with the gauge field part of $G(\omega)$, so we are left with

$$[G(\omega), \psi_f(x)] = \int d^3y\, \omega^a(y)[\psi_g^\dagger(y)T^a\psi_g(y), \psi_f(x)] \tag{69}$$

. Using the identity

$$[\psi_g^\dagger(y)T^a\psi_g(y), \psi_{\beta,f}(x)] = -\delta_{fg}T^a_{\beta\gamma}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\psi_{\gamma,g}(y), \tag{70}$$

we find

$$\begin{aligned}[G(\omega), \psi_{\beta,f}(x)] &= -\int d^3y\, \omega^a(y)\delta_{fg}T^a_{\beta\gamma}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\psi_{\gamma,g}(y) && (71)\\ &= -\omega^a(x)\lambda^a_{\beta\gamma}\psi_{\gamma,f}(x) && (72)\\ &= -\omega(x)\psi(x). && (73)\end{aligned}$$

This is the infinitesimal gauge transformation of $\psi(x)$ in the fundamental representation.

So we see that for all three cases, $G^a(x)$ generates gauge transformation. However, gauge transformations are not physical symmetries and therefore physical states must be invariant under gauge transformations. In order for this to be true, then

$$\begin{aligned}G(x)|\Psi\rangle &= 0, && (74)\\ G(\omega)|\Psi\rangle &= 0. && (75)\end{aligned}$$

This is analagous to Gauss' law in QED.

2.6. With the notation of the previous problem, the Hamiltonian of QCD in the temporal gauge ($A_0 = 0$) reads

$$\mathcal{H} = \int d^3x \left\{ g^2\mathrm{tr}E_k^2(x) + \frac{1}{2g^2}\mathrm{tr}F_{jk}^2(x) + \sum_f \bar{\psi}_f(-i\gamma_k D_k - m_f)\psi_f(x) + \epsilon_0 \right\}, \tag{76}$$

with the following definitions:

$$\begin{aligned}F_{jk} &= \partial_j A_k - \partial_k A_j + i[A_j, A_k], && (77)\\ D_k &= \partial_k + iA_k, && (78)\\ \bar{\psi}_f(x) &= \psi_f^\dagger(x)\gamma^0. && (79)\end{aligned}$$

The additive constant ϵ_0 is chosen in such a way that the vacuum has zero energy.
Show that $[G(x), \mathcal{H}] = 0$, i.e. the Hamiltonian is invariant under gauge transformations.

Solution: The Hamiltonian is constructed entirely from gauge-invariant quantities such as $\mathrm{tr}E^2, \mathrm{tr}F^2, \bar{\psi}\psi$, and therefore any gauge transformation must leave $\mathcal{H}$ invariant by definition.

An alternative argument is using Heisenberg evolution. Physically, this means that Gauss' law is preserved in time. If $|\Psi(t)\rangle$ satisfies $G^a(x)|\Psi(t)\rangle = 0$ at time $t = 0$, then

$$\frac{d}{dt}G^a(\boldsymbol{x}, t) = \frac{i}{\hbar}[\mathcal{H}, G^a(\boldsymbol{x}, t)] = 0 \implies G^a(\boldsymbol{x}, t) = G^a(\boldsymbol{x}, 0). \tag{80}$$

So the constraint is consistent with Hamiltonian evolution, it doesn't break from the dynamics.

2.7. With the notation of the previous two problems, consider the operators

$$Q_f = \int d^3x \, \psi_f^\dagger(x)\psi_f(x), \tag{81}$$

$$U_f(\alpha) = e^{-i\alpha Q_f}. \tag{82}$$

a) Calculate the commutators of Q_f with the fundamental fields $A_k^a(x)$, $E_k^a(x)$, $\psi_g(x)$, and $\psi_g^\dagger(x)$.

b) Calculate

$$U_f(\alpha)A_k^a(x)U_f^\dagger(\alpha), \quad U_f(\alpha)E_k^a(x)U_f^\dagger(\alpha), \quad U_f(\alpha)\psi_g(x)U_f^\dagger(\alpha), \quad U_f(\alpha)\psi_g^\dagger(x)U_f^\dagger(\alpha).$$

c) If P is a generic product of the fundamental fields and their derivatives, relate $[Q_f, P]$ to the number of quark fields of flavour f appearing in the operator P. The operators Q_u and Q_d are called *up-quark number* and *down-quark number* respectively.

d) Show that the Hamiltonian commutes with Q_f. Notice that this implies that $U_f(\alpha)$ is a symmetry of the theory and the corresponding generators Q_f are conserved charges. We say that the *up-quark number* and *down-quark number* are conserved in QCD.

e) Write the baryon-number and electric-charge operators in terms of Q_u and Q_d.

Solution:

a) The gluon $A_k^a(x)$ and chromoelectric $E_k^a(x)$ fields commute with the quark fields and therefore commute with Q_f:

$$[Q_f, A_k^a(x)] = 0, \tag{83}$$

$$[Q_f, E_k^a(x)] = 0. \tag{84}$$

With the anti-commutation relations of quark fields,

$$\{\psi_f(x), \psi_g(y)\} = 0, \tag{85}$$

$$\{\psi_f^\dagger(x), \psi_g(y)\} = -\delta_{fg}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{86}$$

we now compute

$$[Q_f, \psi_g(y)] = \int d^3x \left[\psi_f^\dagger(x)\psi_f(x), \psi_g(y)\right] \tag{87}$$

$$= \int d^3x \left\{\psi_f^\dagger(x)\cancel{[\psi_f(x), \psi_g(y)]} + [\psi_f^\dagger(x), \psi_g(y)]\psi_f(x)\right\} \tag{88}$$

$$= -\int d^3x \, \delta_{fg}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\psi_f(x) \tag{89}$$

$$[Q_f, \psi_g(y)] = -\delta_{fg}\psi_g(y), \tag{90}$$

and similarly

$$[Q_f, \psi_g^\dagger(y)] = +\delta_{fg}\psi_g^\dagger(y). \tag{91}$$

b) Using the BCH formula, we can write

$$U_f X U_f^\dagger = e^{-i\alpha Q_f} X e^{i\alpha Q_f} = e^{-i\alpha[Q_f,\cdot]}X, \quad X \in \{A_k^a(x),\, E_k^a(x),\, \psi_g(x),\, \psi_g^\dagger(x)\}. \tag{92}$$

Since the gluon and chromoelectric fields commute with Q_f, we can see that they are left invariant under U_f:

$$U_f(\alpha)A_k^a(x)U_f^\dagger(\alpha) = A_k^a(x), \tag{93}$$
$$U_f(\alpha)E_k^a(x)U_f^\dagger(\alpha) = E_k^a(x). \tag{94}$$

With the results in eqs. (90) and (91), we can see that the quark fields $\psi_g(x)$ and $\psi_g^\dagger(x)$ will not be invariant under U_f:

$$U_f(\alpha)\psi_g(x)U_f^\dagger(\alpha) = e^{i\alpha\delta_{fg}}\psi_g(x), \tag{95}$$
$$U_f(\alpha)\psi_g^\dagger(x)U_f^\dagger(\alpha) = e^{-i\alpha\delta_{fg}}\psi_g^\dagger(x). \tag{96}$$

c) We recall the Leibniz rule

$$[Q_f, AB] = [Q_f, A]B + A[Q_f, B] \tag{97}$$

which can be performed recursively so it is sufficient to know how Q_f commutes with elementary fields to know how it commutes with a generic product P. As well as the commutators calculated above, we will need that derivatives commute with Q_f, i.e.

$$[Q_f, \partial_\mu\psi_f] = \partial_\mu[Q_f, \psi_f] = -\partial_\mu\psi_f. \tag{98}$$

Let P be a product of fields and derivatives, so it contains n_f factors of ψ_f and m_f factors of $\psi_f^\dagger$. Each ψ_f contributes -1 and each $\psi_f^\dagger$ contributes $+1$ to the commutator, and therefore

$$[Q_f, P] = (m_f - n_f)P. \tag{99}$$

d) Recall the Hamiltonian defined in eq. (76). We can calculate $[Q_f, \mathcal{H}]$ term-by-term. The gauge field terms clearly commute with Q_f, i.e.

$$[Q_f, E^2(x)] = 0, \tag{100}$$
$$[Q_f, F_{jk}^2(x)] = 0. \tag{101}$$

The quark kinetic term commutes as

$$[Q_f, \psi_g^\dagger D_j\psi_g] = [Q_f, \psi_g^\dagger]D_j\psi_g + \psi_g^\dagger D_j[Q_f, \psi_g] = \delta_{fg}\left(\psi_g^\dagger D_j\psi_g - \psi_g^\dagger D_j\psi_g\right) = 0. \tag{102}$$

The quark mass term commutes as

$$[Q_f, \psi_g^\dagger m_g\psi_g] = [Q_f, \psi_g^\dagger]m_g\psi_g + m_g\psi_g^\dagger[Q_f, \psi_g] = \delta_{fg}\left(\psi_g^\dagger m_g\psi_g - \psi_g^\dagger m_g\psi_g\right) = 0. \tag{103}$$

Therefore,

$$[Q_f, \mathcal{H}] = 0. \tag{104}$$

e) Each (anti-)quark carries a baryon number $(-)\frac{1}{3}$, so the total baryon number is

$$B = \frac{1}{3}\sum_f Q_f. \tag{105}$$

Up-type quarks have electric charge $+\frac{2}{3}$ and down-type quarks have electric charge $-\frac{1}{3}$, so the electric-charge operator is

$$Q = \frac{2}{3}Q_u - \frac{1}{3}Q_d. \tag{106}$$