## Lecture 3: One-particle states

3.1. Prove

$$\bar{\boldsymbol{x}}(t) = \bar{\boldsymbol{x}}(0) + \bar{\boldsymbol{v}}\, t, \tag{1}$$
$$\Delta x^2(t) = \Delta x^2(0) + Dt + \Delta v^2 t^2, \tag{2}$$

by using the definition of $f_t(\boldsymbol{x})$,

$$f_t(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}} e^{i\boldsymbol{p}\boldsymbol{x}} \hat{f}_t(\boldsymbol{p}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}} e^{-iE(\boldsymbol{p})t+i\boldsymbol{p}\boldsymbol{x}} \hat{f}(\boldsymbol{p}), \tag{3}$$

and the properties of the Fourier transform and calculate the coefficient $D$ in terms of the wavefunction $\hat{f}(\boldsymbol{p})$.

**Solution:**

a) Starting with

$$\boldsymbol{x}(t) = \int d^3x\, \boldsymbol{x} |f_t(\boldsymbol{x})|^2, \tag{4}$$

we want to calculate the time derivative $\frac{d}{dt}x(t)$ and then integrate it to get the expression for $\bar{\boldsymbol{x}}(t)$. First, using the definition of $f_t(\boldsymbol{x})$, we can rewrite the position-space integral in momentum space.

$$\boldsymbol{x}(t) = \int d^3x\, \boldsymbol{x} \left[ \int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}} \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}} e^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \hat{f}_t^*(\boldsymbol{q}) \hat{f}_t(\boldsymbol{p}) \right] \tag{5}$$
$$= \int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}} \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}} \hat{f}_t^*(\boldsymbol{q}) \hat{f}_t(\boldsymbol{p}) \left[ \int d^3x\, \boldsymbol{x} e^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \right]. \tag{6}$$

This inner $x$ integral is a standard trick in Fourier analysis:

$$\int d^3x\, x_j e^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} = i\frac{\partial}{\partial p_j}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}). \tag{7}$$

So the whole expression becomes

$$\boldsymbol{x}(t) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_t^*(\boldsymbol{p}) (i\nabla_p) \hat{f}_t(\boldsymbol{p}). \tag{8}$$

Now apply the time derivative:

$$\frac{d}{dt}\boldsymbol{x}(t) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \frac{\partial \hat{f}_t^*(\boldsymbol{p})}{\partial t} \cdot (i\nabla_p) \hat{f}_t(\boldsymbol{p}) + \hat{f}_t^*(\boldsymbol{p}) \cdot i\nabla_p \left( \frac{\partial \hat{f}_t(\boldsymbol{p})}{\partial t} \right) \right]. \tag{9}$$

Recall that the momentum-space wavefunction evolves with time as

$$\hat{f}_t(\boldsymbol{p}) = e^{-iE(\boldsymbol{p})t} \hat{f}_0(\boldsymbol{p}). \tag{10}$$

So the time derivatives of the wavefunction are

$$\frac{\partial \hat{f}_t(\boldsymbol{p})}{\partial t} = -iE(\boldsymbol{p})\hat{f}_t(\boldsymbol{p}), \qquad \frac{\partial \hat{f}_t^*(\boldsymbol{p})}{\partial t} = iE(\boldsymbol{p})\hat{f}_t^*(\boldsymbol{p}). \tag{11}$$

Now plug these into the time derivative:

$$\frac{d}{dt}\boldsymbol{x}(t) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \left[ (iE(\boldsymbol{p})\hat{f}_t^*(\boldsymbol{p})) \cdot (i\nabla_p \hat{f}_t(\boldsymbol{p})) + \hat{f}_t^*(\boldsymbol{p}) \cdot i\nabla_p(-iE(\boldsymbol{p})\hat{f}_t(\boldsymbol{p})) \right] \tag{12}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \left[ -E(\boldsymbol{p})\hat{f}_t^*(\boldsymbol{p})\nabla_p \hat{f}_t(\boldsymbol{p}) + |\hat{f}_t(\boldsymbol{p})|^2 \nabla_p E(\boldsymbol{p}) + E(\boldsymbol{p})\hat{f}_t^*(\boldsymbol{p})\nabla_p \hat{f}_t(\boldsymbol{p}) \right]. \tag{13}$$

Notice that the first and third terms cancel each other. What's left is then:

$$\frac{d}{dt}\boldsymbol{x}(t) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} |\hat{f}_t(\boldsymbol{p})|^2 \nabla_p E(\boldsymbol{p}). \tag{14}$$

So what is $\nabla_p E(\boldsymbol{p})$?

$$E(\boldsymbol{p}) = \sqrt{|\mathbf{p}|^2 + m^2} \implies \nabla_p E(\boldsymbol{p}) = \frac{\boldsymbol{p}}{E(\boldsymbol{p})}. \tag{15}$$

Substituting this in:

$$\frac{d}{dt}\boldsymbol{x}(t) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} |\hat{f}_t(\boldsymbol{p})|^2 \frac{\boldsymbol{p}}{E(\boldsymbol{p})} = \bar{\boldsymbol{v}}(t). \tag{16}$$

[**Note:** of course, it can be sufficient simply to identify the velocity $\bar{\boldsymbol{v}}(t)$ and integrate.] Finally integrating, we get

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(0) + \bar{\boldsymbol{v}}t. \tag{17}$$

b) Starting from

$$\Delta x^2(t) = \int d^3x \, [\boldsymbol{x} - \bar{\boldsymbol{x}}(t)]^2 |f_t(\boldsymbol{x})|^2, \tag{18}$$

we can expand this to see what must be quantified:

$$\Delta x^2(t) = \int d^3x \, \boldsymbol{x}^2 |f_t(\boldsymbol{x})|^2 - 2\bar{\boldsymbol{x}}(t) \int d^3x \, \boldsymbol{x} |f_t(\boldsymbol{x})|^2 + \bar{\boldsymbol{x}}(t)^2 \int d^3x \, |f_t(\boldsymbol{x})|^2 \tag{19}$$

$$= \langle x^2 \rangle_t - \bar{\boldsymbol{x}}(t)^2. \tag{20}$$

We already have

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(0) + \bar{\boldsymbol{v}}t \implies \bar{\boldsymbol{x}}(t)^2 = \boldsymbol{x}(0)^2 + 2t\boldsymbol{x}(0)\bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}^2 t^2, \tag{21}$$

so we need to calculate $\langle x^2 \rangle_t$. In momentum space, the position operator is

$$\boldsymbol{x} = i\nabla_p \implies \boldsymbol{x}^2 = -\nabla_p^2, \tag{22}$$

so

$$\langle \boldsymbol{x}^2 \rangle(t) = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_t^*(\boldsymbol{p})(-\nabla_p^2)\hat{f}_t(\boldsymbol{p}). \tag{23}$$

Applying the derivative to the wavefunction, we find

$$\nabla_p \hat{f}_t(\boldsymbol{p}) = \nabla_p(\hat{f}_t(\boldsymbol{p})) = \nabla_p \left( e^{-iE(\boldsymbol{p})t} \hat{f}_0(\boldsymbol{p}) \right) \tag{24}$$

$$= e^{-iE(\boldsymbol{p})t} \left[ -it(\nabla_p E(\boldsymbol{p}))\hat{f}_0(\boldsymbol{p}) + \nabla \hat{f}_0(\boldsymbol{p}) \right] \tag{25}$$

$$\nabla_p^2 \hat{f}_t(\boldsymbol{p}) = e^{-iE(\boldsymbol{p})t} \Big[ -t^2(\nabla_p E(\boldsymbol{p}))^2 \hat{f}_0(\boldsymbol{p}) - 2it\nabla_p E(\boldsymbol{p}) \cdot \nabla_p \hat{f}_0(\boldsymbol{p}) - it(\nabla_p^2 E(\boldsymbol{p}))\hat{f}_0(\boldsymbol{p}) + \nabla_p^2 \hat{f}_0(\boldsymbol{p}) \Big] \tag{26}$$

Now insert this result into eq. (23):

$$\langle \boldsymbol{x}^2\rangle(t) = -\int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} e^{iE(\boldsymbol{p})t} \hat{f}_0^*(\boldsymbol{p}) e^{-iE(\boldsymbol{p})t} \times \Big[ -t^2(\nabla_p E(\boldsymbol{p}))^2 \hat{f}_0(\boldsymbol{p}) - 2it\nabla_p E(\boldsymbol{p}) \cdot \nabla_p \hat{f}_0(\boldsymbol{p}) - it(\nabla_p^2 E(\boldsymbol{p}))\hat{f}_0(\boldsymbol{p}) + \nabla_p^2 \hat{f}_0(\boldsymbol{p}) \Big] \tag{27}$$

Let's take this by powers of $t$:

- $t^0$ term:

$$\langle \boldsymbol{x}^2\rangle(t) \supset \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_0^*(\boldsymbol{p})(-\nabla_p^2 \hat{f}_0(\boldsymbol{p})) = \langle x^2\rangle(0) \tag{28}$$

- $t^1$ term:

$$\langle \boldsymbol{x}^2\rangle(t) \supset -2it \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_0^* \nabla_p \hat{f}_0 \cdot \nabla_p E(\boldsymbol{p}) \tag{29}$$

- $t^2$ term:

$$\langle \boldsymbol{x}^2\rangle(t) \supset t^2 \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} |\hat{f}_0(\boldsymbol{p})|^2 (\nabla_p E(\boldsymbol{p}))^2 = t^2 \langle v^2\rangle \tag{30}$$

By subtracting this with eq. (21), we find

$$\Delta x^2(t) = \langle \boldsymbol{x}^2\rangle(t) - \bar{\boldsymbol{x}}(t)^2 \tag{31}$$

$$= (\langle \boldsymbol{x}^2\rangle(0) - \bar{\boldsymbol{x}}(0)^2) + Dt + (\langle v^2\rangle - \bar{\bar{v}}^2)t^2 \tag{32}$$

$$= \Delta x^2(0) + Dt + \Delta v^2 t^2, \tag{33}$$

and we can write $D$ as

$$D = -2i \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_0^*(\boldsymbol{p}) \nabla_p \hat{f}_0(\boldsymbol{p}) \cdot \nabla_p E(\boldsymbol{p}) - 2\bar{\boldsymbol{x}}(0) \cdot \bar{\boldsymbol{v}} \tag{34}$$

$$= -2i \int \frac{2^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}_0^*(\boldsymbol{p}) \nabla_p \hat{f}_0(\boldsymbol{p}) \cdot (\nabla_p E(\boldsymbol{p}) - \bar{\boldsymbol{v}}), \tag{35}$$

where we can combine the terms since we recognise they share the expression for $\bar{\boldsymbol{x}}(0)$, however there remains a difference because $\bar{\boldsymbol{x}}(t)^2$ uses the *mean velocity* $\bar{\boldsymbol{v}}$, while $\langle \boldsymbol{x}^2\rangle(t)$ uses the *local velocity* $\nabla_p E(\boldsymbol{p})$.

In $\langle x^2\rangle(t)$, each momentum mode carries its own velocity, while in $\bar{\boldsymbol{x}}(t)^2$, the motion is summarised by the average group velocity.

3.2. Calculate the action of the Poincaré group on $\tilde{\phi}(p)$ starting from

$$U(\Lambda, a)\phi_(x)U(\Lambda, a)^{-1} = R_{nm}(\Lambda^{-1})\phi_m(\Lambda x + a). \tag{36}$$

**Solution:** Recall the definition of the Fourier-transformed field:

$$\tilde{\phi}(p) = \int d^4x\, e^{ipx}\, \phi(x). \tag{37}$$

Now applying the action of the Poincaré group,

$$U(\Lambda,a)\phi_(p)U(\Lambda,a)^{-1} = \int d^4x\, e^{ipx} U(\Lambda,a)\phi_(x)U(\Lambda,a)^{-1} \tag{38}$$

$$= \int d^4x\, e^{ipx} \phi(\Lambda^{-1}(x-a)). \tag{39}$$

Now if we change variables: $y = \Lambda^{-1}(x-a)$, $x = \Lambda y + a$, $d^4x = d^4y$, then

$$U(\Lambda,a)\phi_(p)U(\Lambda,a)^{-1} = \int d^4y\, e^{ip(\Lambda y+a)}\phi(y) \tag{40}$$

$$= e^{ipa}\int d^4y\, e^{i(\Lambda^{-1}p)\cdot y}\phi(y) \tag{41}$$

$$U(\Lambda,a)\phi_(p)U(\Lambda,a)^{-1} = e^{ipa}\tilde{\phi}(\Lambda^{-1}p). \tag{42}$$

3.3. Prove that

$$e^{-iHt}a(f)^\dagger|\Omega\rangle = a(f_t)^\dagger|\Omega\rangle. \tag{43}$$

**Solution:** With the creation operator defined

$$a(f)^\dagger|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}(\boldsymbol{p})|p\rangle = |f\rangle, \tag{44}$$

then

$$e^{-iHt}a(f)^\dagger|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} e^{-iHt}\hat{f}(\boldsymbol{p})|p\rangle \tag{45}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}_t(\boldsymbol{p})|p\rangle \tag{46}$$

$$= a(f_t)^\dagger|\Omega\rangle. \tag{47}$$

3.4. Using arguments similar to those in section 3.2.3, show that the operator $a(f)$ annihilates the vacuum, i.e.

$$a(f)|\Omega\rangle = 0. \tag{48}$$

**Solution:** The annihilation operator can be written

$$a(f) = \frac{1}{Z^{1/2}}\int \frac{d^4p}{(2\pi)^4}F(p)\tilde{\phi}(p). \tag{49}$$

So as was the strategy for the creation operator, we need to know how the integrand acts on the vacuum:

$$F(p)\tilde{\phi}(p)|\Omega\rangle = \mathbb{P}_1 F(p)\tilde{\phi}(p)|\Omega\rangle \tag{50}$$

$$= F(p)\int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}|q\rangle\langle q|\phi(p)|\Omega\rangle \tag{51}$$

$$= F(p)\int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\int d^4x\, e^{ipx}|q\rangle\langle q|\phi(x)|\Omega\rangle \tag{52}$$

$$= F(p)\int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}(2\pi)^4\delta(p_0+E(\boldsymbol{q}))\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}). \tag{53}$$

Note that this result will project onto a state $|{-q}\rangle$ which has negative energy, however the function $F(p)$ requires that states be on the positive mass shell, so Therefore,

$$F(p)\tilde{\phi}(p)|\Omega\rangle = 0. \tag{54}$$

3.5. Calculate the matrix element $\langle \boldsymbol{p}|\phi^{\chi}(x)^{\dagger}|\Omega\rangle$ for the smeared field $\phi^{\chi}(x)$, where $\chi(x)$ is defined

$$\tilde{\chi}(p)^{*} = \sqrt{2E(\boldsymbol{p})}\,\tilde{\zeta}(p_0 - E(\boldsymbol{p}))\,\tilde{\eta}(\boldsymbol{p}). \tag{55}$$

**Solution:** Reminder of the smeared field:

$$\phi^{\chi}(x)^{\dagger} = \int d^4y\,\chi^{*}(x-y)\phi(y)^{\dagger}. \tag{56}$$

The matrix element can be written

$$\langle p|\phi^{\chi}(x)^{\dagger}|0\rangle = \int d^4y\,\chi^{*}(x-y)\langle p|\phi(y)^{\dagger}|\Omega\rangle, \tag{57}$$

where in the lectures we saw that

$$\langle p|\phi(y)^{\dagger}|\Omega\rangle = Z^{1/2}e^{ipy}. \tag{58}$$

So then

$$\langle p|\phi^{\chi}(x)^{\dagger}|0\rangle = Z^{1/2}\int d^4y\,\chi^{*}(x-y)\,e^{ipx}. \tag{59}$$

If we recognise the complex conjugate of the Fourier transform of $\chi$ as

$$\tilde{\chi}(q)^{*} = \int d^4z\,\chi^{*}(z)e^{-iqz} \implies \int d^4y\,\chi^{*}(x-y)e^{\pm iqy} = e^{\pm iqx}\tilde{\chi}^{*}(\pm q), \tag{60}$$

then

$$\langle p|\phi^{\chi}(x)^{\dagger}|0\rangle = Z^{1/2}\chi^{*}(p)e^{ipx}. \tag{61}$$

3.6. Check that, in the case of the free theory, the creation operator $a(f)^{\dagger}$ defined

$$\begin{aligned} a(f)^{\dagger} &= \frac{1}{Z^{1/2}}\int \frac{d^4p}{(2\pi)^4}\,F(p)\tilde{\phi}^{\dagger}(p) \\ &= \frac{1}{Z^{1/2}}\int \frac{d^4p}{(2\pi)^4}\,\hat{f}(\boldsymbol{p})\tilde{\zeta}(p_0 - E(\boldsymbol{p}))\,\tilde{\phi}^{\dagger}(p), \end{aligned} \tag{62}$$

coincides with the creation operator defined in the free theory integrated against the wave-function $\hat{f}(\boldsymbol{p})$.

**Solution:** Recall the scalar field,

$$\phi(x) = \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\left[a(\boldsymbol{q})e^{-iE(\boldsymbol{p})x_0 + i\boldsymbol{q}\boldsymbol{x}} + a(\boldsymbol{q})^{\dagger}e^{iE(\boldsymbol{q})x_0 - i\boldsymbol{q}\boldsymbol{x}}\right], \tag{63}$$

which is Hermitian, i.e. $\phi^\dagger(x) = \phi(x)$. The conjugate of the Fourier-transformed field is

$$\tilde{\phi}(p)^\dagger = \int d^4x e^{-ipx} \phi^\dagger(x) \tag{64}$$
$$= \int d^4x e^{-ipx} \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})} \left[a(\boldsymbol{q})^\dagger e^{iE(\boldsymbol{q})x_0 - i\boldsymbol{q}\boldsymbol{x}} + a(\boldsymbol{q}) e^{-iE(\boldsymbol{q})x_0 + i\boldsymbol{q}\boldsymbol{x}}\right] \tag{65}$$
$$= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})} \int d^4x \left[a(\boldsymbol{q})^\dagger e^{i(E(\boldsymbol{q})-p_0)x_0} e^{i(\boldsymbol{q}-\boldsymbol{p})\boldsymbol{x}} + a(\boldsymbol{q}) e^{i(E(\boldsymbol{q})+p_0)} e^{i(\boldsymbol{q}-\boldsymbol{p})\boldsymbol{x}}\right], \tag{66}$$

and now performing the position integral yields delta functions:

$$= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})} \Big[a(\boldsymbol{q})^\dagger\, 2\pi\delta(p_0 - E(\boldsymbol{q}))\, (2\pi)^3 \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) + a(\boldsymbol{q})\, 2\pi\delta(p_0 + E(\boldsymbol{q}))\, (2\pi)^3 \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p})\Big] \tag{67}$$

and then performing the momentum integral yields

$$= \frac{2\pi}{2E(\boldsymbol{p})} \Big[a(\boldsymbol{p})^\dagger \delta(p_0 - E(\boldsymbol{p})) + a(\boldsymbol{p})\, \delta(p_0 + E(\boldsymbol{p}))\Big]. \tag{68}$$

Now we can insert this into $a(f)^\dagger$:

$$a(f)^\dagger = \frac{1}{Z^{1/2}} \int \frac{d^4p}{(2\pi)^4} \hat{f}(\boldsymbol{p}) \tilde{\zeta}(p_0 - E(\boldsymbol{p}))\, \frac{2\pi}{2E(\boldsymbol{p})} \Big[a(\boldsymbol{p})^\dagger \delta(p_0 - E(\boldsymbol{p})) + a(\boldsymbol{p})\, \delta(p_0 + E(\boldsymbol{p}))\Big]. \tag{69}$$

The $\tilde{\zeta}(p_0 - E(\boldsymbol{p}))$ function selects only states on the positive mass shell, so what does it do to our result above? Consider the $p_0$ integrals:

$$\int dp_0\, \tilde{\zeta}(p_0 - E(\boldsymbol{p}))\delta(p_0 - E(\boldsymbol{p})) = 1, \tag{70}$$
$$\int dp_0\, \tilde{\zeta}(p_0 - E(\boldsymbol{p}))\delta(p_0 + E(\boldsymbol{p})) = 0. \tag{71}$$

So $\tilde{\zeta}$ cuts off the contribution which would yield negative energy. We then find

$$a(f)^\dagger = \frac{1}{Z^{1/2}} \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})} \hat{f}(\boldsymbol{p})\, a^\dagger(\boldsymbol{p}). \tag{72}$$

3.7. Consider the operator

$$A_t(f) = \frac{1}{Z^{1/2}} \int d^3x\, f_t(\boldsymbol{x})\, \phi^\chi(t, \boldsymbol{x})^\dagger. \tag{73}$$

Check that, in the case of the free theory, $A_t(f)$ does not depend on $t$.

**Solution:** First, we remind ourselves of the full definition of $f_t(\boldsymbol{x})$:

$$f_t(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E(\boldsymbol{p})}} e^{i\boldsymbol{p}\boldsymbol{x}} \hat{f}_t(\boldsymbol{p}) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E(\boldsymbol{p})}} e^{-iE(\boldsymbol{p})t + i\boldsymbol{p}\boldsymbol{x}} \hat{f}(\boldsymbol{p}). \tag{74}$$

Then, inserting the standard definition for $\phi$ into the smeared field, we find

$$\phi^\chi(t,\boldsymbol{x})^\dagger = \int d^4y\,\chi^*(x-y)\int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\left[a(\boldsymbol{q})e^{-iE(\boldsymbol{q})t'+i\boldsymbol{q}\boldsymbol{y}} + a(\boldsymbol{q})^\dagger e^{iE(\boldsymbol{q})t'-i\boldsymbol{q}\boldsymbol{y}}\right] \tag{75}$$

$$= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\int d^4y\,\chi^*(x-y)\left[a(\boldsymbol{q})e^{-iE(\boldsymbol{q})t'+i\boldsymbol{q}\boldsymbol{y}} + a(\boldsymbol{q})^\dagger e^{iE(\boldsymbol{q})t'-i\boldsymbol{q}\boldsymbol{y}}\right]. \tag{76}$$

Recalling eq. (60) for the Fourier transform of $\chi^*$, we have

$$\phi^\chi(t,\boldsymbol{x})^\dagger = \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\left[a(\boldsymbol{q})e^{-iE(\boldsymbol{q})t+i\boldsymbol{q}\boldsymbol{x}}\tilde{\chi}^*(-q) + a(\boldsymbol{q})^\dagger e^{iE(\boldsymbol{q})t-i\boldsymbol{q}\boldsymbol{x}}\tilde{\chi}^*(q)\right]. \tag{77}$$

We know that

$$\tilde{\chi}^*(q) = \sqrt{2E(\boldsymbol{q})}\tilde{\zeta}(q_0 - E(\boldsymbol{q}))\tilde{\eta}(\boldsymbol{q}). \tag{78}$$

This selects only the positive energy components, which is got from the $a(\boldsymbol{q})^\dagger$ term, so

$$\phi^\chi(t,\boldsymbol{x})^\dagger = \int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}}\,a(\boldsymbol{q})^\dagger e^{iE(\boldsymbol{q})t-i\boldsymbol{q}\boldsymbol{x}}\tilde{\eta}(\boldsymbol{q}). \tag{79}$$

Now inserting back into $A_t(f)$:

$$A_t(f) = \frac{1}{Z^{1/2}}\int d^3x\, f_t(\boldsymbol{x})\,\phi^\chi(t,\boldsymbol{x})^\dagger \tag{80}$$

$$= \frac{1}{Z^{1/2}}\int d^3x\int \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}}e^{-iE(\boldsymbol{p})t+i\boldsymbol{p}\boldsymbol{x}}\hat{f}(\boldsymbol{p})\int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}}\,a(\boldsymbol{q})^\dagger\tilde{\eta}(\boldsymbol{q})e^{iE(\boldsymbol{q})t-i\boldsymbol{q}\boldsymbol{x}} \tag{81}$$

$$= \frac{1}{Z^{1/2}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}}\hat{f}(\boldsymbol{p})\int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}}\,a(\boldsymbol{q})^\dagger\tilde{\eta}(\boldsymbol{q})\int d^3x\, e^{i(E(\boldsymbol{q})-E(\boldsymbol{q}))t-i(\boldsymbol{q}-\boldsymbol{p})\boldsymbol{x}} \tag{82}$$

$$= \frac{1}{Z^{1/2}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E(\boldsymbol{p})}}\hat{f}(\boldsymbol{p})\int \frac{d^3q}{(2\pi)^3\sqrt{2E(\boldsymbol{q})}}\,a(\boldsymbol{q})^\dagger\tilde{\eta}(\boldsymbol{q})\,(2\pi)^3\,\delta^{(3)}(\boldsymbol{q}-\boldsymbol{p})\,e^{iE(\boldsymbol{q})t-iE(\boldsymbol{p})t} \tag{83}$$

$$= \frac{1}{Z^{1/2}}\int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}(\boldsymbol{p})\,a(\boldsymbol{p})^\dagger\tilde{\eta}(\boldsymbol{p}), \tag{84}$$

where we see the time dependence drops out.