Lecture 4: Asymptotic multi-particle states

4.1. Let $\hat{f}^{\alpha}(\boldsymbol{p})$ be the wavefunctions considered in this chapter. Calculate the following connected 2-point functions

$$\langle A_t(f^\alpha)\, A_t(f^\beta)^\dagger \rangle_c \tag{1}$$

for any combination of $\alpha, \beta = 1, 2$, in terms of the momentum-space wavefunctions $\hat{f}^{\alpha}(\boldsymbol{p})$.

Solution: We saw that

$$A_t(f^\beta)^\dagger|\Omega\rangle = e^{iHt}a(f_t^\beta)^\dagger|\Omega\rangle = e^{iHt}|f_t^\beta\rangle = |f^\beta\rangle, \tag{2}$$

$$|f^\beta\rangle = \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}^\beta(\boldsymbol{p})|p\rangle. \tag{3}$$

With the conjugate $\langle 0|A_t(f^\alpha) = \langle f^\alpha|$, then

$$\begin{aligned}
\langle A_t(f^\alpha)\, A_t(f^\beta)^\dagger \rangle_c &= \langle f^\alpha|f^\beta\rangle \quad (4)\\
&= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*\alpha}(\boldsymbol{q}) \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}^\beta(\boldsymbol{p})\,\langle q|p\rangle \quad (5)\\
&= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*\alpha}(\boldsymbol{q}) \int \frac{d^3p}{(2\pi)^3 2E(\boldsymbol{p})}\hat{f}^\beta(\boldsymbol{p})\,(2\pi)^3 2E(\boldsymbol{q})\,\delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) \quad (6)\\
&= \int \frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*\alpha}(\boldsymbol{q})\,\hat{f}^\beta(\boldsymbol{q}). \quad (7)
\end{aligned}$$

4.2. Let $\hat{f}^{\alpha}(\boldsymbol{p})$ be the wavefunctions considered in this chapter. Derive the cluster decomposition of the following 4-point function:

$$\langle\Omega|A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger|\Omega\rangle, \tag{8}$$

identifying which terms vanish identically. Use the results of Problem 4.1 to simplify your answer.

Solution: The full decomposition follows section 4.3.2:

$$\begin{aligned}
&\langle\Omega|A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger|\Omega\rangle\\
&= \langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)A_t(f^2)\rangle_c\langle A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c + \langle A_t(f^2)\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^2)^\dagger\rangle_c\langle A_t(f^1)A_t(f^2)A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)^\dagger\rangle_c\langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)\rangle_c\langle A_t(f^2)\rangle_c\langle A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)\rangle_c\langle A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)\rangle_c\langle A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c + \langle A_t(f^2)\rangle_c\langle A_t(f^2)^\dagger\rangle_c\langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^2)\rangle_c\langle A_t(f^1)^\dagger\rangle_c\langle A_t(f^1)A_t(f^2)^\dagger\rangle_c + \langle A_t(f^2)^\dagger\rangle_c\langle A_t(f^1)^\dagger\rangle_c\langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)\rangle_c\langle A_t(f^2)\rangle_c\langle A_t(f^2)^\dagger\rangle_c\langle A_t(f^1)^\dagger\rangle_c.
\end{aligned} \tag{9}$$

We know that all 1-point functions $\langle A_t(f^i)^{(\dagger)}\rangle_c = 0$ so we are left with

$$\begin{aligned}
&\langle\Omega|A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger|\Omega\rangle\\
&= \langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)A_t(f^2)\rangle_c\langle A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c\\
&\quad + \langle A_t(f^1)A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c + \langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c.
\end{aligned} \tag{10}$$

Since $A_t(f^i)$ annihilates the vacuum, any n-point function with one of these on the right is also zero, i.e. $\langle A_t(f^1)A_t(f^2)\rangle_c = 0$. We are left with:

$$\begin{aligned}&\langle\Omega|A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger|\Omega\rangle\\&=\langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c\\&\quad+\langle A_t(f^1)A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c+\langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c.\end{aligned}\tag{11}$$

We can see the form of the connected 2-point functions in eq. (7), and so we have

$$\langle A_t(f^1)A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c=\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*1}(\boldsymbol{q})\,\hat{f}^2(\boldsymbol{q})\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*2}(\boldsymbol{q})\,\hat{f}^1(\boldsymbol{q}),\tag{12}$$

$$\langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c=\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*1}(\boldsymbol{q})\,\hat{f}^1(\boldsymbol{q})\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*2}(\boldsymbol{q})\,\hat{f}^2(\boldsymbol{q}),\tag{13}$$

and since we assume $\hat{f}^\alpha$ are the wavefunctions discussed through the lectures, we know that they are normalised to unity and orthonormal, i.e.

$$\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*1}(\boldsymbol{q})\,\hat{f}^2(\boldsymbol{q})=0\implies\langle A_t(f^1)A_t(f^2)^\dagger\rangle_c\langle A_t(f^2)A_t(f^1)^\dagger\rangle_c=0\tag{14}$$

$$\int\frac{d^3q}{(2\pi)^3 2E(\boldsymbol{q})}\hat{f}^{*1}(\boldsymbol{q})\,\hat{f}^1(\boldsymbol{q})=1\implies\langle A_t(f^1)A_t(f^1)^\dagger\rangle_c\langle A_t(f^2)A_t(f^2)^\dagger\rangle_c=1.\tag{15}$$

The connected 4-point function can be written as

$$\begin{aligned}\langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c&=\frac{1}{Z^2}\int d^3x_1\,d^3x_2\,d^3x_3\,d^3x_4\,f_t^{1*}(\boldsymbol{x}_4)\,f_t^{2*}(\boldsymbol{x}_3)\,f_t^2(\boldsymbol{x}_2)\,f_t^1(\boldsymbol{x}_1)\\&\qquad\times\langle\phi_1^\chi(t,\boldsymbol{x}_4)\phi_2^\chi(t,\boldsymbol{x}_3)\phi_2^\chi(t,\boldsymbol{x}_2)^\dagger\phi_3^\chi(t,\boldsymbol{x}_1)^\dagger\rangle_c\end{aligned}\tag{16}$$

$$\langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle_c=\frac{1}{Z^2}I^{4\mathrm{pt-conn}}(t).\tag{17}$$

The total 4-point function is therefore

$$\langle A_t(f^1)A_t(f^2)A_t(f^2)^\dagger A_t(f^1)^\dagger\rangle=1+\frac{1}{Z^2}I^{4\mathrm{pt-conn}}(t).\tag{18}$$

4.3. Let $\hat{f}^\alpha(\boldsymbol{p})$ be the wavefunctions considered in this chapter. Calculate the $t\to+\infty$ limit of the 4-point function in Problem 4.2, and show that the limit is approached with an error that vanishes rapidly.

Solution: We can see in eq. (18) that it is the 4-point connected piece which is dependent on time and so we need to take the $t\to\infty$ limit for $I^{4\mathrm{pt-conn}}(t)$. Following Ruelle's cluster theorem, we can write that the Wightman function

$$\langle\phi_1^\chi(t,\boldsymbol{x}_1)\phi_2^\chi(t,\boldsymbol{x}_1)\phi_2^\chi(t,\boldsymbol{x}_3)^\dagger\phi_3^\chi(t,\boldsymbol{x}_4)^\dagger\rangle_c$$

in $I^{4\mathrm{pt-conn}}(t)$ is bounded

$$|W(\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3)|\le\frac{B_r}{(1+|\boldsymbol{z}_2|)^r(1+|\boldsymbol{z}_3|)^r(1+|\boldsymbol{z}_4|)^r},\tag{19}$$

where we have transformed the variables

$$\boldsymbol{z}_\alpha = \boldsymbol{x}_\alpha - \boldsymbol{x}_1. \tag{20}$$

Following the implications of this, we will find

$$|I(t)| \leq C_r \int d^3x_1\, d^3x_2 \frac{|f_t^2(\boldsymbol{x}_2) f_t^1(\boldsymbol{x}_1)|}{(1 + |\boldsymbol{x}_2 - \boldsymbol{x}_1|)^r} \equiv J(t). \tag{21}$$

Now it is better to define the velocity-space wavefunctions

$$\mathring{f}(\boldsymbol{v}) = \begin{cases} t^{3/2} e^{im\gamma^{-1}t} f_t(t\boldsymbol{v}), & |\boldsymbol{v}| < 1, \\ t^{3/2} f_t(t\boldsymbol{v}), & |\boldsymbol{v}| \geq 1. \end{cases} \tag{22}$$

Changing variables to velocity, (21) becomes

$$J(t) = C_r \int d^3v_1\, d^3v_2 \frac{|\mathring{h}_t^2(\boldsymbol{v}_2) \mathring{h}_t^1|}{(1 + t|\boldsymbol{v}_2 - \boldsymbol{v}_1|)^r}. \tag{23}$$

Following **Theorem 4.4**, we define the open sets U_1, U_2 containing the sets of velocities $V(f^1)$, $V(f^2)$, and we can separate the integration domain into four pieces,

$$J(t) = J(t; U_1, U_2) + J(t; U_1, U_2^c) + J(t; U_1^c, U_2) + J(t; U_1^c, U_2^c), \tag{24}$$

and then each sub-integral vanishes asymptotically:

$$J(t; U_1^c, U_2^c) \leq C_r D_{1,r} D_{2,r} \left\{ \int \frac{d^3v}{(1 + |\boldsymbol{v}|)^r} \right\}^2 t^{-2r}, \tag{25}$$

$$J(t; U_1^c, U_2) \leq D_{1,r} D_2' \left\{ \int \frac{d^3v}{(1 + |\boldsymbol{v}|)^3} \right\}^2 t^{3/2-r}, \tag{26}$$

$$J(t; U_1, U_2^c) \leq D_1' D_{2,r} \left\{ \int \frac{d^3v}{(1 + |\boldsymbol{v}|)^3} \right\}^2 t^{3/2-r}, \tag{27}$$

$$J(t; U_1, U_2) \leq C_r D_1' D_2'\, d(U_1, U_2)^{-r}\, t^{3-r} \left\{ \int \frac{d^3v}{(1 + |\boldsymbol{v}|)^r} \right\}^2. \tag{28}$$